Graphons

- We introduce graphons to study graph filters and GNNs in the limit of large number of nodes

Definition (Graphon)

A graphon is a bounded symmetric measurable function $\Rightarrow \mathrm{W} : [0,1]^2 \to [0,1]$

- Can think of a graphon as a weighted symmetric graph with uncountable nodes

$\Rightarrow$ The labels are the graphon arguments $\Rightarrow u \in [0,1]$.

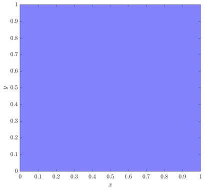
$\Rightarrow$ The weights are the graphon values $\Rightarrow W(u,v) = W(v,u)$

Graphon Examples

Definition (Graphon)

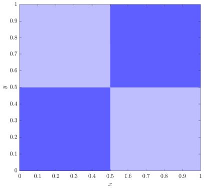
A graphon is a bounded symmetric measurable function $\Rightarrow W : [0,1]^2 \rightarrow [0,1]$

Uniform (Erdős-Rényi)



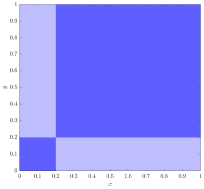
$W(u,v) = p$

Balanced stochastic block model (SBM)



$W(u,v) = p \gg W(u,v) = q$

Unbalanced (SBM)



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The Purpose of a Graphon

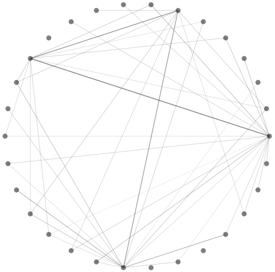
Definition (Graphon)

A graphon is a bounded symmetric measurable function $\Rightarrow W : [0,1]^2 \rightarrow [0,1]$

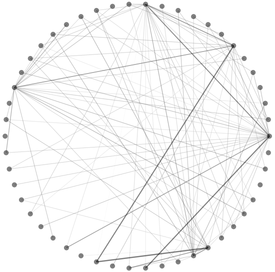
- Practice $\Rightarrow$ Graph sets where graphs in the set have large number of nodes and similar structure
- Theory $\Rightarrow$ A generative model of graph families via deterministic or stochastic edge sampling
- Theory $\Rightarrow$ A limit object for a sequence of graphs

The Product Similarity “Graphon”

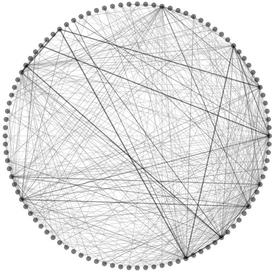
- Product similarity graphs, even with different number of nodes, “look like each other”
- Abstract similarities between graphs into a limit object $\Rightarrow$ The product similarity “graphon”



$n = 30$ products



$n = 50$ products

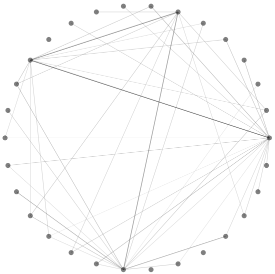


$n = 100$ products

The Product Similarity "Graphon"

- We never compute the product similarity "graphon"

$\Rightarrow$ Use abstract idea of graphon to work with all of these graphs as if they were the same object



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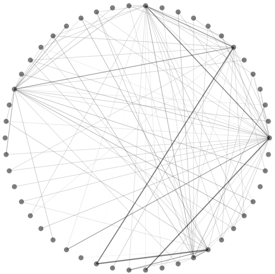

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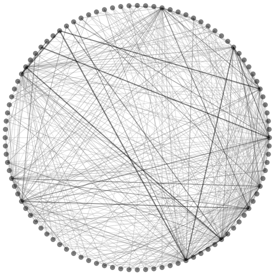


$n = 100$ products

- Vertices: For an n-node graph, sample n points $\{u_1, u_2, \ldots, u_n\}$ from the unit interval $[0, 1]$

  $\Rightarrow$ Points can be sampled on a grid, uniformly at random, etc.

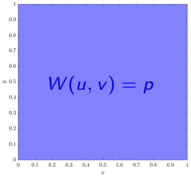
  $\Rightarrow$ Each sample u_i corresponds to a node $i \in \{1, 2, 3, \ldots, n\}$ of the graph

- Edges: Evaluate $\mathrm{W}(u_i, u_j)$ for edge (i, j)

  $\Rightarrow$ Stochastic: Connect i and j with an unweighted undirected edge with probability $\mathrm{W}(u_i, u_j)$

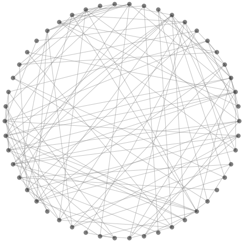
  $\Rightarrow$ Weighted: Connect i and j with weighted undirected edge with weight $\mathrm{W}(u_i, u_j)$

- Use uniform Graphon



To generate random graphs with the same

Or different number of nodes



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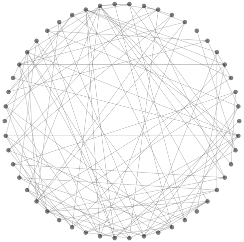

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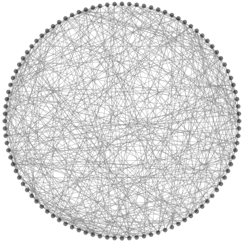
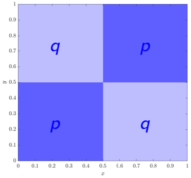


$n = 100$ nodes

Balanced SBM Graphons as Generative Models

- Use balanced SBM

  Graphon



To generate balanced SBM graphs with the same

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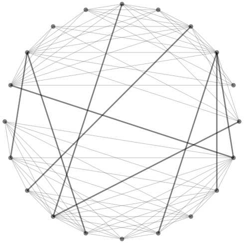
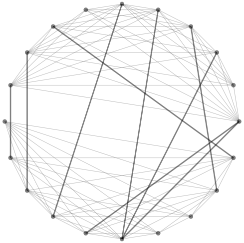


$n = 20$ nodes



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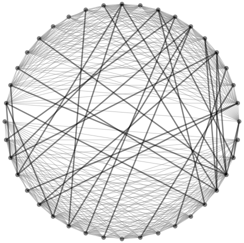
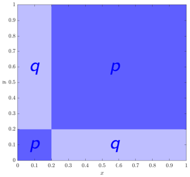


$n = 40$ nodes

- Use Unbalanced SBM Graphon



To generate unbalanced SBM graphs with the same

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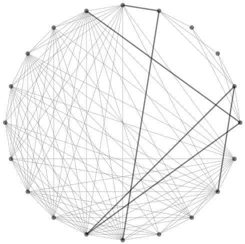

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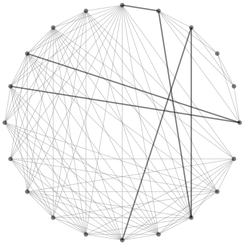
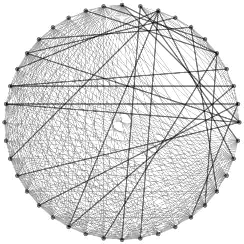


$n = 20$ nodes

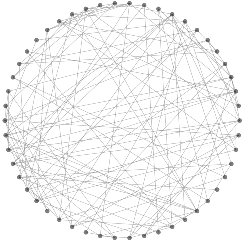


$n = 40$ nodes

Uniform Graphons as a Limit Object

- As we consider random graphs with larger numbers of nodes the graphs approach a limit

  $\Rightarrow$ It is unclear what that limit is. The graphon is the limit. As we will see



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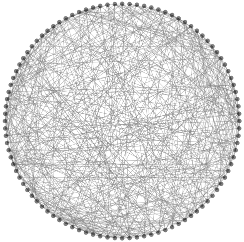

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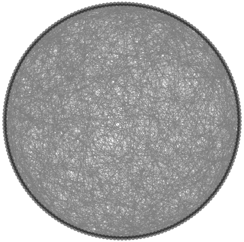

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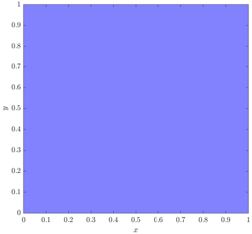


Graphon $W(u, v) = p$

Convergence of Graph Sequences

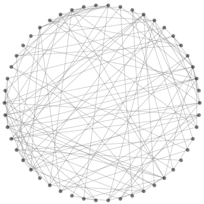
- A graphon is the limit of a sequence of graphs that converges in terms of homomorphism densities

Convergent Graph Sequences

- Sequence of graphs with growing number of nodes n $\Rightarrow \left\{G_n = (V_n, E_n, S_n)\right\}_{n=1}^{\infty}$.

- The graph sequence $\{G_n\}_{n=1}^{\infty}$ converges to a graphon W $\Rightarrow$ In what sense?

  $\Rightarrow$ We need to introduce three concepts: Motifs, homomorphisms, and homomorphism densities



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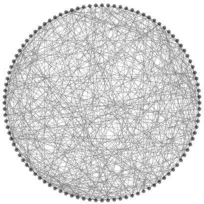

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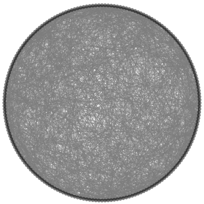

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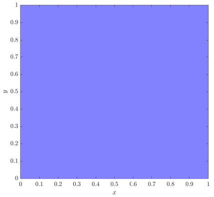
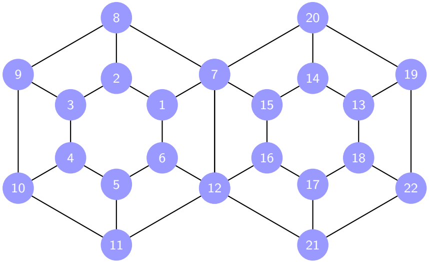


Graphon $W(u, v) = p$

- A motif F is a graph. But think of it as a small graph that we embed in another larger graph



- Homomorphisms are adjacency preserving maps from motif $F = (V', E')$ into graph $G = (V, E)$

$$\beta \;:\; V' \;\to\; V \;\text{ such that }\; \Big(i, j\Big) \;\in\; E' \;\text{ implies }\; \Big(\beta(i), \beta(j)\Big) \;\in\; E$$

Motifs and Graph Homomorphisms

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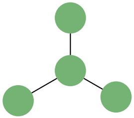

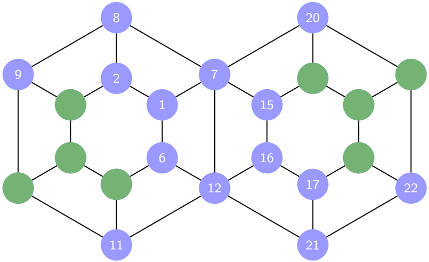

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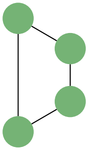
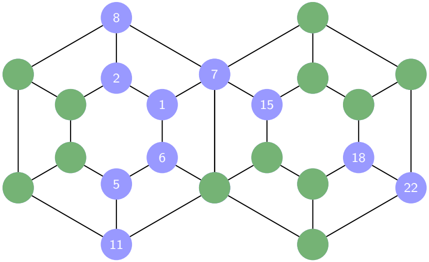

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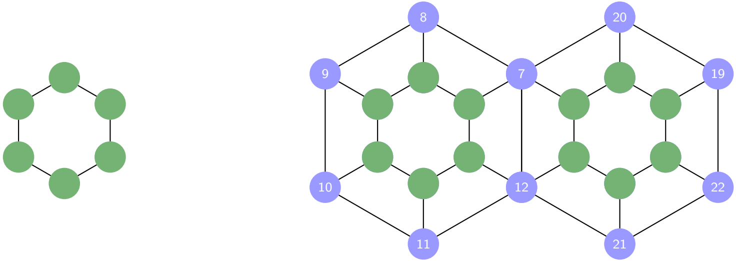

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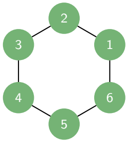

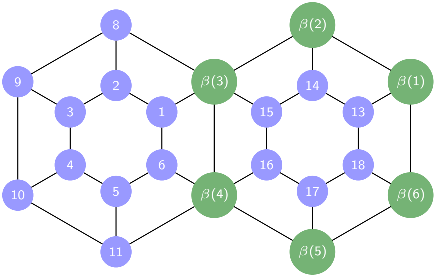

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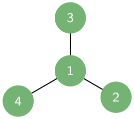
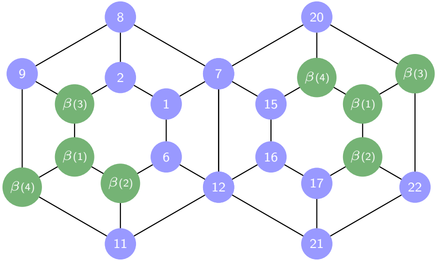
- Given motif F and graph G, there are multiple homomorphism functions β





- We define $\text{hom}(F, G)$ to represent the number of homomorphisms between motif F and graph G

- If the graph G has n nodes and the motif F has n' nodes, there are $n^{n'}$ different maps from F to G

- Homomorphism density of motif F in graph G is the fraction of maps that are homomorphisms

$$t(F, G) = \frac{\text{hom}(F, G)}{n^{n'}}$$

- Density $t(F, G)$ is a relative measure of the number of ways in in which F can be mapped into G

- Consider weighted graph $G = (V, E, S)$ with adjacency matrix S

- Homomorphism density of motif F in weighted graph G with the adjacency matrix S is

$$t(F, G) \;=\; = \frac{\sum_{\beta} \prod_{(i,j)\in\mathcal{E}'} [\mathsf{S}]_{\beta(i)\beta(j)}}{n^{n'}}$$

- Weight each motif embedding by the product of the edge weights in the homomorphism image.

Homomorphism Density for Graphons

- The Homomorphism density of a motif F into a given graphon W is defined as

$$t(F, W) = \int_{[0,1]^{n'}} \prod_{(i,j)\in\mathcal{E}'} W(u_i, u_j) \prod_{i\in\mathcal{V}'} du_i$$

- The homomorphism density is the probability of drawing the motif F from the graphon W

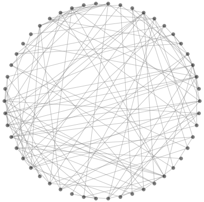
Definition (Convergent graph sequence)

A sequence of undirected graphs G_n converges to the graphon W if and only if for all motifs F

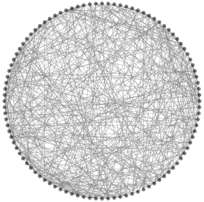
$$\lim_{n\to\infty} t(F, G_n) = t(F, W)$$

- We say that the sequence G_n converges to W in the homomorphism density sense
- It can be proven that every graphon is the limit object of a sequence of convergent graphs
- It can be proven that every convergent graph sequence converges to a graphon

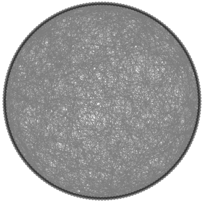
- Consider a sequence of random graphs $\{G_n\}$ sampled from the graphon W. Graphs G_n have
  - $\Rightarrow$ Labels $u_i \sim U[0,1]$ drawn uniformly at random from the interval $[0,1]$
  - $\Rightarrow$ Edge sets such that $(u_i, u_j) \in \mathcal{E}$ with probability $W(u_i, u_j)$
- We have $\lim_{n\to\infty} t(F, G_n) = t(F, W)$ in the homomorphism density sense almost surely



$n = 50$ nodes $\rightarrow$



$n = 100$ nodes $\rightarrow$



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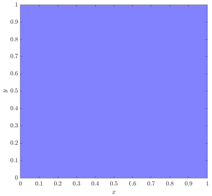
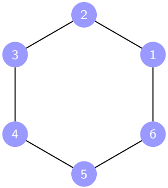


Graphon $W(u, v) = p$

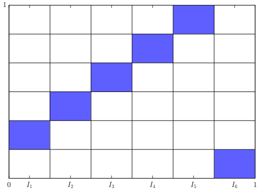
Induced Graphons

- Every undirected graph admits a graphon representation which we call its induced graphon
- Consider a graph $G = \{\mathcal{V}, \mathcal{E}, \mathbf{S}\}$ with $|\mathcal{V}| = n$ and normalized graph shift operator $\mathbf{S}$
- Regular partition of the unit interval with n subintervals $\Rightarrow \mathrm{I}_i = \Big[(i-1)/n,\ i/n\Big)$
- We define the induced graphon W_G $\Rightarrow W_G(u,v) = [\mathbf{S}]_{ij}\ \mathbb{I}(u \in \mathrm{I}_i)\ \mathbb{I}(v \in \mathrm{I}_j)$



Cycle graph G with $n = 6$ nodes

$\rightarrow$



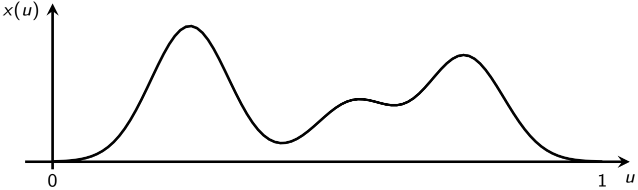
Graphon W_G induced by the graph G

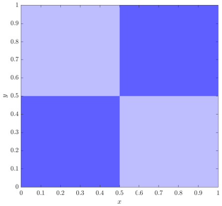
Graphon Signals

- Graph signals are signals supported on graphons. They are limit objects of graph signals

- Graphon signals are pairs (W, X) where W is a graphon and $X : [0, 1] \to \mathbb{R}$ is a function

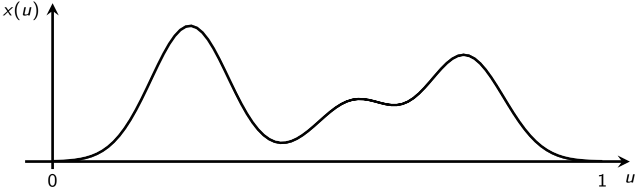
- Function $X(u) \in L^2([0, 1])$ has finite energy $\Rightarrow \int_0^1 |X(u)|^2 du < \infty$.

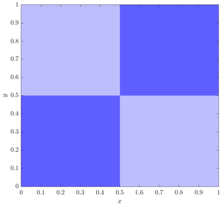




- Generative models of graph signals. And limits of convergent sequences of graph signals

- We generate graph signals $(\mathbf{S}_n, \mathbf{x}_n)$ by taking n samples of the graphon signal (W, X)
- Sample the graphon at node labels u_i. Sample the function X at node labels u_i $\Rightarrow x_i = X(u_i)$
- Graph signal sampled from the unit interval in the same coordinates u_i where graphon is sampled





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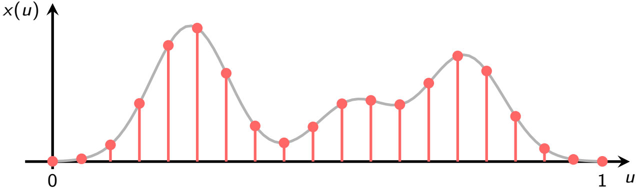
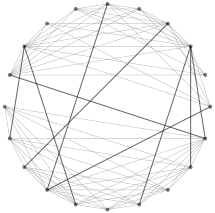
- Graph signal sampled from the unit interval in the same coordinates u_i where graphon is sampled

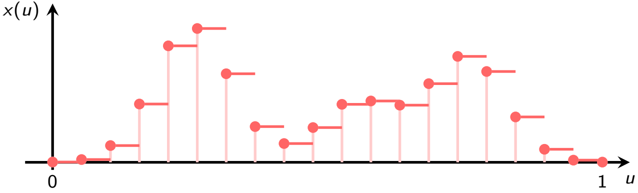


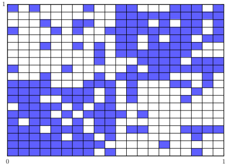


- Every graph signal x supported on graph G induces a graphon signal (W_G, X_G)
- Regular partition of unit interval with n subintervals $\mathrm{I}_i = \left[(i-1)/n,\ i/n\right)$

$\Rightarrow$ Induced signal $X_G(u) = x_i\,\mathbb{I}(u \in I_i)$

$\Rightarrow$ W_G is the graphon induced by the graph G $\Rightarrow$ $W_G(u,v) = [\mathrm{S}]_{ij}\,\mathbb{I}(u \in \mathrm{I}_i)\,\mathbb{I}(v \in \mathrm{I}_j)$





Definition (Convergent sequences of graph signals)

A sequence of graph signals $(G_n, \mathbf{x}_n)$ is said to converge to the graphon signal (W, X), if there exists a sequence of permutations π_n such that for all motifs F we have

$$t(F, G_n) \to t(F, W), \qquad \text{and} \left\| X_{\pi_n(G_n)} - X \right\|_{L^2} \to 0$$

We say (W, X) is the limit of the graph signal sequence and write $(G_n, x_n) \to (W, X)$

- The permutation is used here to make the convergence definition independent of labels
- To enable comparison of the vector $\mathbf{x}_n$ and the function X we use the induced signal in the L_2 norm

- The Graphon W can be used to define an integral linear operator $\Rightarrow T_W : L^2([0,1]) \to L^2([0,1])$

- When applied to the graphon signal X, the operator T_W produces the signal $T_W X$ with values

$$(T_W X)(v) = \int_0^1 W(u,v) X(u)\, du$$

- This is a Hilbert-Schmidt operator because W is bounded and compact. It's a matrix multiplication

- We say that the linear operator T_W is the graphon shift operator (WSO) of the graphon W

$\Rightarrow$ Applying the WSO T_W to the graphon signal X diffuses X over the graphon W

Graphon Fourier Transform

- We define a graphon Fourier transform to enable spectral representation of graphon signals.

- The WSO is a self adjoint Hilbert-Schmidt operator $\Rightarrow (T_{\mathrm{W}}X)(v) = \int_0^1 \mathrm{W}(u, v)X(u)\, du$

- The function $\varphi : [0, 1] \to \mathbb{R}$ is an eigenfunction of T_W with associated eigenvalue λ if

$$(T_{\mathrm{W}}\varphi)(v) \;=\; \int_0^1 \mathrm{W}(u, v)\varphi(u)\, du \;=\; \lambda\varphi(v)$$

- T_{W} has a countable number of eigenvalue-eigenfunction pairs $\Rightarrow \Big\{ (\lambda_i, \varphi_i) \Big\}_{i=1}^{\infty}$

- We assume eigenfunctions are normalized to unit energy $\Rightarrow \|\varphi_i\|^2 = \int_0^1 \varphi(u)du = 1$

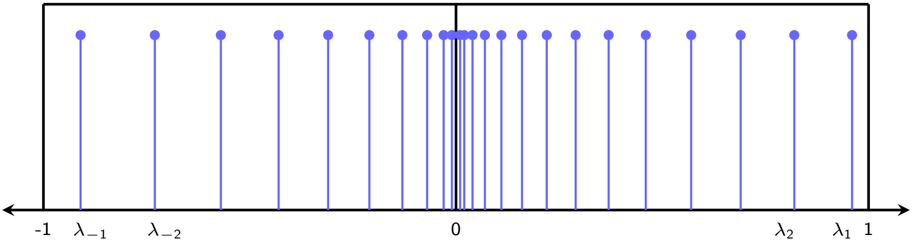
- The (countable number of) eigenfunctions of the operator T_w are an orthonormal basis of $L^2\big([0,1]\big)$

- We can thus decompose the graphon W in the basis $\{\varphi_i\}_{i=1}^{\infty}$ of eigenfunctions of the operator T_W

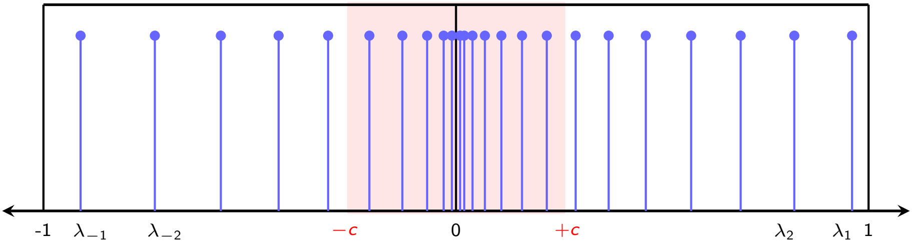
$$\mathsf{W}(u,v) \;=\; \sum_{i=0}^{\infty} \lambda_i \, \varphi_i(u)\, \varphi_i(v)$$

- More or less the same as the eigenvector decomposition $\Rightarrow \mathsf{S} \;=\; \mathsf{V}\Lambda\mathsf{V}^H \;=\; \sum_{i=0}^{\infty} \lambda_i \, \mathsf{v}_i \, \mathsf{v}_i^T$

- T_W is self adjoint and $0 \leq W(x,y) \leq 1 \Rightarrow$ Eigenvalues are real and lie in the interval $[-1, 1]$
- Order them as $\Rightarrow -1 \leq \lambda_{-1} \leq \lambda_{-2} \leq \ldots \leq 0 \leq \ldots \leq \lambda_2 \leq \lambda_1 \leq 1$



- Graphon eigenvalues accumulate at $\lambda = 0$ $\Rightarrow \lim_{i\to\infty} \lambda_i = \lim_{i\to\infty} \lambda_{-i} = 0$. And only at $\lambda = 0$
- For any $c > 0$, the number of eigenvalues with $|\lambda_i| \geq c$ is finite $\Rightarrow \#\left\{\lambda_i : |\lambda_i| \geq c\right\} = n_c < \infty$
- All eigenvalues that are not $\lambda_j = 0$ have finite multiplicity



Theorem (Eigenvalue Convergence of a Graph Sequence)

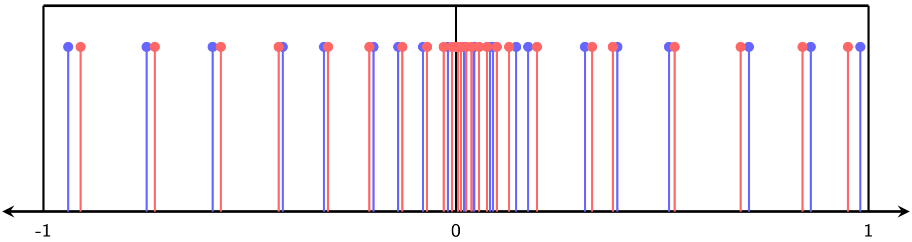
If a graph sequence $\{\mathrm{G}_n\}$ converges to a graphon W in the homomorphism density sense , then

$$\lim_{n\to\infty} \frac{\lambda_j(\mathrm{S}_n)}{n} = \lambda_j(T_{\mathrm{W}}) = \lim_{n\to\infty} \lambda_j(T_{\mathrm{W}_n}) \text{ for all } j$$

- For any convergent graph sequence, the eigenvalues of the graph converge to those of the graphon

Borgs-Chayes-Lovász-Sós-Vesztergombi, *Convergent Sequences of Dense Graphs II. Multiway Cuts and Statistical Physics,*

- For a convergent graph sequence, eigenvalues of the graph converge to those of the limit graphon



- Convergence holds in the sense that $\Rightarrow \exists\, n_0$ s.t. for all $n > n_0$, $\left|\dfrac{\lambda_j(\mathbf{S}_n)}{n} - \lambda_j(T_{\mathbf{W}})\right| < \epsilon\,, \epsilon > 0$
- But n_0 will be different for each j. Eigenvalue convergence is not uniform

The Graphon Shift Operator Induces a Transform

- The graphon shift operator can be rewritten as

$$(T_W\phi)(v) = \sum_{j=0}^{\infty} \lambda_j \varphi_j(v) \int_0^1 \varphi_j(u) X(u) du$$

- Integral terms correspond to inner products $\langle X, \varphi_j \rangle$ between the signal and the eigenfunctions
- Moreover, the eigenfunctions form a complete orthonormal basis of $L^2([0, 1])$
- Thus, the inner products can provide a complete representation of the signal on the graphon basis
- That change of basis is called the graphon Fourier Transform

Definition (Graphon Fourier transform)

The graphon Fourier transform (WFT) of a graphon signal X is defined as a functional $\hat{X} =$ WFT(X) with continuous input X and discrete output

$$\hat{X}_j = \hat{X}(\lambda_j) = \int_0^1 X(u)\varphi_j(u)du$$

with $\{\lambda_j\}_{j \in \mathbb{Z}/\{0\}}$ the eigenvalues and $\{\varphi_j\}_{j \in \mathbb{Z}/\{0\}}$ the eigenfunctions of T_{W}

- The eigenvalues λ_j are countable $\Rightarrow$ The graphon Fourier transform $\hat{X}$ can always be defined

Definition (Inverse graphon Fourier transform)

The inverse graphon Fourier transform (iWFT) of a graphon Fourier transform $\hat{X}$ is defined as

$$\text{iWFT}(\hat{X}) = \sum_{j \in \mathbb{Z}/\{0\}} \hat{X}(\lambda_j)\varphi_j = X$$

with $\{\lambda_j\}_{j \in \mathbb{Z}/\{0\}}$ the eigenvalues and $\{\varphi_j\}_{j \in \mathbb{Z}/\{0\}}$ the eigenfunctions of T_W

- Eigenfunctions $\{\varphi_j\}_{j \in \mathbb{Z}/\{0\}}$ are orthonormal. The iWFT is a proper inverse of the WFT

The GFT converges to the WFT

- We discuss the convergence of the GFT to the WFT for graph sequences that converge to graphons.
- This need us to review convergence of eigenvectors and eigenvalues of graph sequences

The Graphon Fourier Transform and the Graph Fourier Transform

- Graphon FT, $\text{WFT}(W, X)$ is the eigenspace projection $\Rightarrow \hat{X}_j = \hat{X}(\lambda_j) = \int_0^1 X(u)\, \varphi_j(u)\, du$

- Graph FTs, $\text{GFT}(G_n, \mathbf{x}_n)$ are the eigenspace projections $\Rightarrow \hat{x}_n(j) = \hat{x}_n(\lambda_{nj}) = \sum_{i=1}^{n} x_n(i)\, \mathbf{v}_{nj}(i)$

- Graph signal sequence $(G_n, \mathbf{x}_n)$ converges to graphon signal (W, X) $\Rightarrow$ Conjecture GFT convergence

$$\text{GFT}(G_n, \mathbf{x}_n) \;\to\; \text{WFT}(W, X)$$

- Eigenvalue convergence holds $\Rightarrow \lambda_{nj} \to \lambda_j$. Conjecture is reasonable GFT convergence should hold

- Graphon FT, $\text{WFT}(W, X)$ is the eigenspace projection $\Rightarrow \hat{X}_j = \hat{X}(\lambda_j) = \int_0^1 X(u)\, \varphi_j(u)\, du$

- Graph FTs, $\text{GFT}(G_n, \mathrm{x}_n)$ are the eigenspace projections $\Rightarrow \hat{x}_n(j) = \hat{x}_n(\lambda_{nj}) = \sum_{i=1}^{n} x_n(i)\, \mathrm{v}_{nj}(i)$

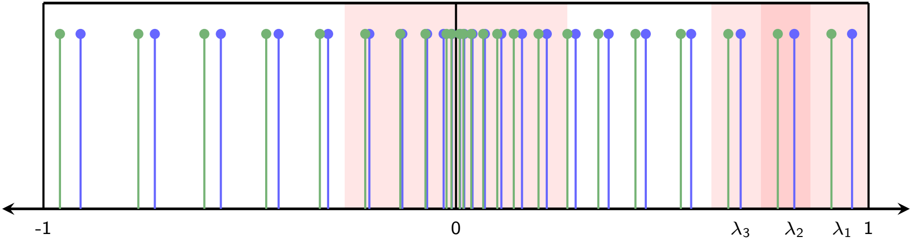
- Alas, this conjecture is wrong $\Rightarrow$ GFT convergence to the WFT does not hold in general

$$\text{GFT}(G_n, \mathrm{x}_n) \not\to \text{WFT}(W, X)$$

- GFT and WFT are projections on eigenvectors and eigenfunctions. Not eigenvalues

Convergence to Graphon Eigenvectors

- Convergence of two eigenvectors depends on how close the eigenvalues of other eigenvectors are
- Eigenvalues accumulate around $\lambda = 0$. They all converge. But different eigenvalues are close
- It makes the eigenvectors slow to converge $\Rightarrow$ They all converge but convergence is not uniform



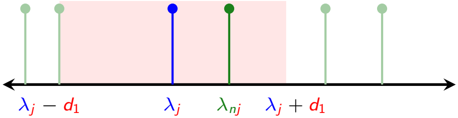
- Consider eigenvalues λ_j of graphon W and λ_{nj} of graph G_n with the same index j

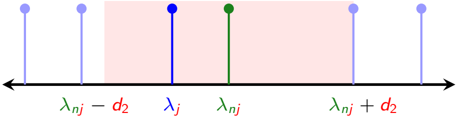
$\Rightarrow$ Compare graphon eigenvalue λ_j to the closest graph eigenvalue other than λ_{nj}

$\Rightarrow$ Compare graph eigenvalue λ_{ni} to the closest graphon eigenvalue other than λ_j

$$d(\lambda_j, \lambda_{nj}) \; = \; \min \left(d_1 \; = \; \min_{i \neq j} \Big| \lambda_j - \lambda_{ni} \Big|, \; d_2 \; = \; \min_{i \neq j} \Big| \lambda_{nj} - \lambda_i \Big| \right)$$

$\Rightarrow$ The minimum of these two is the eigenvalue margin $d(\lambda_j, \lambda_{nj})$ for the eigenvalue pair $(\lambda_j, \lambda_{nj})$





Theorem (Davis-Kahan)

Given graphon W and graphon W_{G_n} induced by graph G_n we consider graphon eigenvalue λ_j and graph eigenvalue λ_{nj}. The distance between the associated eigenfunctions is bounded by

$$\|\varphi_j - \varphi_{nj}\| \leq \frac{\pi}{2} \frac{\|W - W_{G_n}\|}{d(\lambda_j, \lambda_{nj})}$$

where $d(\lambda_j, \lambda_{nj})$ is the eigenvalue margin for the eigenvalue pair $(\lambda_j, \lambda_{nj})$

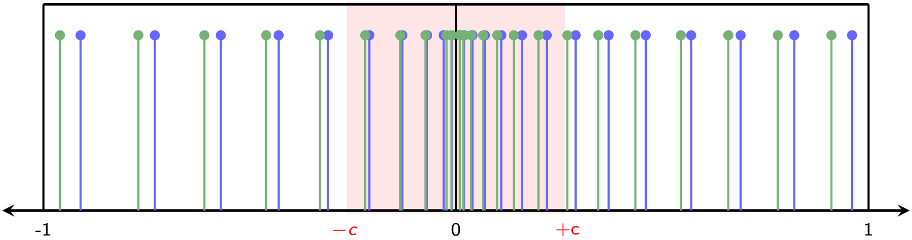
- Graph eigenvectors converge to graphon eigenfunctions if graph sequence converges to graphon
- When the distance to other eigenvalues decreases, the distance between eigenvectors increases

The GFT Does Not Converge to the WFT

- For eigenvalues close to 0 the margin $d(\lambda_j, \lambda_{nj})$ vanishes $\Rightarrow$ There are infinite eigenvalues in $[-c, c]$

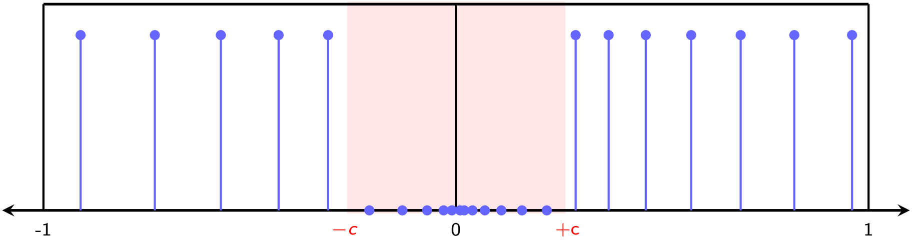
- Thus for any n and $\epsilon > 0$ we have some j for which $\Rightarrow \dfrac{\pi}{2} \dfrac{\|W - G_n\|}{d(\lambda_j, \lambda_{nj})} > \epsilon$

- Opposite of a convergence claim. $\Rightarrow$ For any $\epsilon > 0$, all $n > n_0$, and j $\Rightarrow \dfrac{\pi}{2} \dfrac{\|W - G_n\|}{d(\lambda_j, \lambda_{nj})} \leq \epsilon$



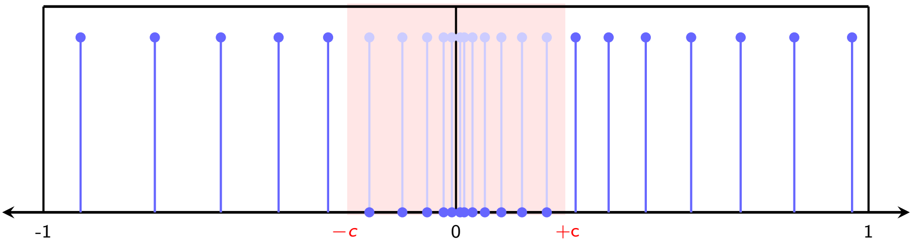
Definition (Graphon bandlimited signals)

A graphon signal (W, X) is c-bandlimited, with bandwith $c \in (0, 1]$, if $\hat{X}(\lambda_j) = 0$ for all $|\lambda_j| < c$.



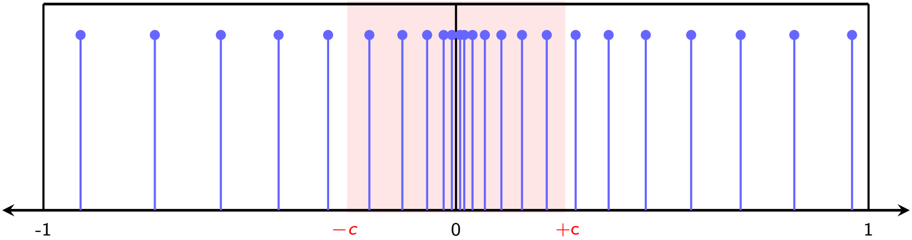
- Just to emphasize the simplicity of this definition consider a graphon signal that is Not-Bandlimited

- To make it bandlimited it suffices for us to nullify all of the WFT components in the interval $(-c, c)$



Bandlimited and Not-Bandlimited Graphon Signals

- Just to emphasize the simplicity of this definition consider a graphon signal that is Not-Bandlimited
- To make it bandlimited it suffices for us to nullify all of the WFT components in the interval $(-c, c)$



Theorem (GFT convergence for graphon bandlimited signals)

Let $(G_n, \mathbf{x}_n)$ be a sequence of graph signals converging to the c-bandlimited graphon signal (W, X).

There exists a sequence of permutations π_n such that

$$\text{GFT}\Big(\pi_n(G_n), \pi_n(\mathbf{x}_n)\Big) \to \text{WFT}\Big(W, X\Big)$$

Proof: See course webpage https://gnn.seas.upenn.edu/lectures/lecture-9/ ■

Theorem (iGFT convergence for graphon bandlimited signals)

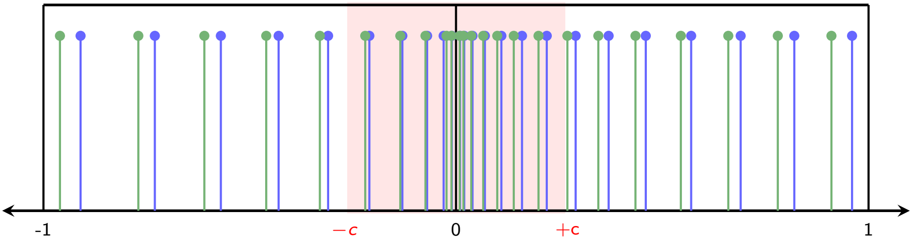
Let $(G_n, \hat{\mathbf{x}}_n)$ be a sequence of GFTs converging to the WFT (W, X). The WFT is associated to a c-bandlimited graphon signal. There exists a sequence of permutations $\{\pi_n\}$ such that

$$\pi_n\Big(\text{iGFT}(\hat{\mathbf{x}}_n)\Big) \rightarrow \text{iWFT}\Big(\hat{X}\Big).$$

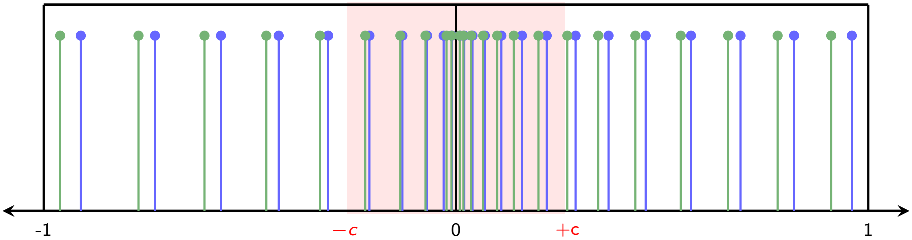
Proof: See course webpage https://gnn.seas.upenn.edu/lectures/lecture-9/ ■

- Convergence of GFT depends on convergence of graph eigenvalues to graphon eigenvalues

- As the number of nodes n grows, the eigenvalues of G_n converge to the eigenvalues of W.



- However, for large $|j|$ the graph and graphon eigenvalues become difficult to tell apart
- Therefore, the GFT only converges to the WFT for graphon bandlimited signals



Graphon Filters

- We define graphon filters and prove their frequency response, which is independent of the graphon.

- Apply the Graphon shift operator recursively to create the graphon diffusion sequence

$$\left(T_{\mathrm{W}}^{(k)} X \right)(v) \;=\; \int_0^1 \mathrm{W}(u,v) \left(T_{\mathrm{W}}^{(k-1)} X \right)(u)\, du \qquad T_{\mathrm{W}}^{(0)} X = X$$

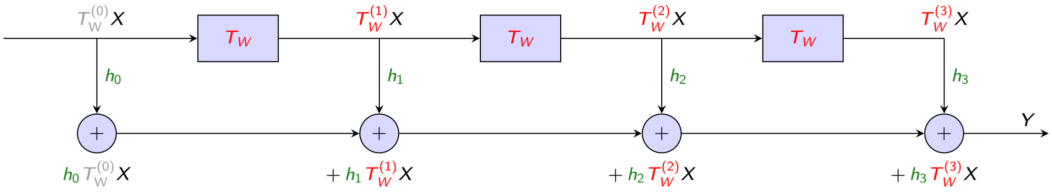
- A graphon filter of order K is defined by the filter coefficients h_k and produces outputs as per

$$Y(v) \;=\; \sum_{k=1}^{K} h_k \left(T_{\mathrm{W}}^{(k)} X \right)(v) \;=\; (T_H X)(v)$$

- A linear combination of the elements of the diffusion sequence modulated by coefficients h_k

- A graphon filter has the same algebraic structure of a graph filter $\Rightarrow Y(v) = \sum_{k=1}^{K} h_k \left(T_W^{(k)} X \right)(v)$

- Only difference is a change of shift operator $\Rightarrow T_W X \ : \ (T_W)X(v) \ = \ \int_0^1 W(u,v)\, X(u)\, du$



$\Rightarrow$ WFTs of input signal $\Rightarrow \hat{X}_j = \int_0^1 X(u)\varphi_j(u)du$ $\Rightarrow$ WFT of output $\Rightarrow \hat{Y}_j = \int_0^1 Y(u)\varphi_j(u)du$

Theorem (Graph frequency representation of graphon filters)

Given a graphon filter T_{H} with coefficients h_k, the components of the graphon Fourier transforms of the input and output signals are related by

$$\hat{Y}_j = \sum_{k=0}^{K} h_k \lambda_j^k \hat{X}_j$$

- The same polynomial that defines the filter but with the eigenvalue λ_i as a variable

Proof: See course webpage https://gnn.seas.upenn.edu/lectures/lecture-9/ ∎

- Graphon filters are pointwise in the WFT domain $\Rightarrow \hat{Y}_j = \sum_{k=0}^{K} h_k \lambda_j^k \hat{X}_j = h(\lambda_j)\,\hat{X}_j$

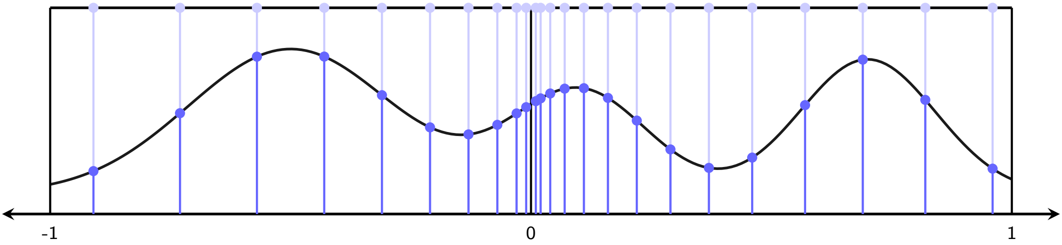
Definition (Frequency response of a graphon filter)

Given a graphon filter with coefficients $\mathrm{h} = \{h_k\}_{k=1}^{\infty}$ the frequency response is the polynomial

$$h(\lambda) = \sum_{k=0}^{\infty} h_k \lambda^k$$

- This is also the exact same definition of the frequency response of a graph filter with coefficients h_k

- The frequency response of a graphon filter and a graph filter with the same coefficients are the same
- Graphon filter instantiates graphon eigenvalues. Graph filter instantiates graph eigenvalues
- If graph sequence converges to a graphon eigenvalues converge $\Rightarrow$ The filter transfers



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