Graphon Filters

1 Frequency Response of Graphon Filters

Theorem 1 (Frequency response of graphon filters) *The WFT of input graphon signal is $\hat{X}_j = \int_0^1 X(u)\varphi_j(u)du$, while WFT of output graphon signal is $\hat{Y}_j = \int_0^1 Y(u)\varphi_j(u)du$. Given a graphon filter $T_{\mathbf{H}}$ with coefficients h_k, the components of the graphon Fourier transforms of the input and output signals are related by*

$$\hat{Y}_j \;=\; \sum_{k=0}^{K} h_k \lambda_j^k \hat{X}_j \tag{1}$$

Proof: Based on the decomposition of $\mathbf{W}(u,v)$, operator $T_{\mathbf{W}}$ can be rewritten as:

$$(T_{\mathbf{W}}X)(v) = \sum_{j=0}^{\infty} \lambda_j \varphi_j(v) \int_0^1 \varphi_j(u) X(u) du = \sum_{j=0}^{\infty} \lambda_j \varphi_j(v) \hat{X}_j. \tag{2}$$

Therefore,

$$\begin{aligned}(T_{\mathbf{W}}{}^{(2)}X)(v) &= \int_0^1 \sum_{i=0}^{\infty} \lambda_i \varphi_i(u) \varphi_i(v) \sum_{j=0}^{\infty} \lambda_j \varphi_j(u) \hat{X}_j du \\ &= \int_0^1 \sum_{j=0}^{\infty} \lambda_j{}^2 \varphi_j(v) \hat{X}_j du = \sum_{j=0}^{\infty} \lambda_j{}^2 \varphi_j(v) \hat{X}_j.\end{aligned}$$

The summation disappears because of the orthonormality of eigenfunctions φ and only the term $i = j$ is left. By repeating the above process ecursively, we can get that:

$$(T_{\mathbf{W}}{}^{(k)}X)(v) = \sum_{j=0}^{\infty} \lambda_j{}^k \varphi_j(v) \hat{X}_j.$$

The output graphon signal therefore can be rewritten as:

$$Y(v) = \sum_{j \in \mathbb{Z} \setminus \{0\}} \sum_{k=0}^{K} h_k \lambda_j^k \varphi_j(v) \hat{X}_j.$$

WFT of $Y(v)$ is:

$$\hat{Y}_j = \int_0^1 \sum_{j \in \mathbb{Z} \setminus \{0\}} \sum_{k=0}^{K} h_k \lambda_j^k \varphi_j(v) \hat{X}_j \varphi_j(u) du = \sum_{k=0}^{K} h_k \lambda_j^k \hat{X}_j.$$

This concludes our proof. ■