Convergence of the GFT to the WFT

Alejandro Ribeiro, Luana Ruiz and Luiz Chamon

1 Assumptions and preliminary results

Before showing formally how and why the GFT of sequences of graphs converge to the WFT the limit, we state clearly our assumptions and we introduce some basic theorems, corollaries and lemmas necessary for our discussion.

1.1 Assumptions

Assumption 1 The graphons induced by graphs and the graphons considered to be the limit of a sequence of graphs are *nonderogatory* (see definition below).

Definition 1 *A graphon* $\mathbf{W}$ *is non-derogatory if* $\lambda_i \neq \lambda_j$ *for all* $i \neq j$ *and* $i, j \in \mathbb{Z} \setminus \{0\}$.

Assumption 2 The graphon signals induced by graph signals and the graphon signal limits are *bandlimited* (see definition below).

Definition 2 (Graphon bandlimited signals) *A graphon signal* $(\mathbf{W}, X)$ *is c-bandlimited, with bandwith* $c \in [0,1]$*, if* $\hat{X}(\lambda_j) = 0$ *for all* j *such that* $|\lambda_j| < c$.

Assumption 3 Although the GSOs $\mathbf{S}_n$ of the graphs $\mathbf{G}_n$ have a finite number of eigenvalues $\lambda_j(\mathbf{S}_n)$, we still associate the eigenvalue sign with its index and order the eigenvalues in decreasing order of absolute value. The indices j are now defined on some finite set $\mathcal{L} \subseteq \mathbb{Z} \setminus \{0\}$.

Assumption 4 Taking into account that the induced graphon representations are defined uniquely from the sequences of graphs and graph signals, we use in our analysis these induced representations. In particular, we will consider the graphon $\mathbf{W}_{\mathbf{G}_n}$ induced by $\mathbf{G}_n$ and the graphon signal $X_{\mathbf{G}_n}$ induced by $\mathbf{x}_n$.

1.2 Preliminary results

In the following lemma we provide the formal connection between a graph signal and its induced graphon representation.

Lemma 1 *Let* $(\mathbf{W}_\mathbf{G}, X_\mathbf{G})$ *be the graphon signal induced by the graph signal* $(\mathbf{G}, \mathbf{x})$ *on* n *nodes. Then, for* $j \in \mathcal{L}$ *we have*

$$\lambda_j(T_{\mathbf{W}_\mathbf{G}}) = \frac{\lambda_j(\mathbf{S})}{n}$$
$$\varphi_j(T_{\mathbf{W}_\mathbf{G}})(u) = [\mathbf{v}_j]_k \times \sqrt{n}\mathbb{I}\,(u \in I_k)$$
$$[\hat{X}_\mathbf{G}]_j = \frac{[\hat{\mathbf{x}}]_j}{\sqrt{n}}$$

where $\lambda_j(\mathbf{S})$ *are the eigenvalues of the graph. For* $j \notin \mathcal{L}$, $\lambda_j(T_{\mathbf{W}_\mathbf{G}}) = [\hat{X}_\mathbf{G}]_j = 0$ *and* $\varphi_j(T_{\mathbf{W}_\mathbf{G}}) = \psi_j$ *such that* $\{\varphi_j(T_{\mathbf{W}_\mathbf{G}})\} \cup \{\psi_j\}$ *forms an orthonormal basis of* $L^2([0,1])$.

Proof: Refer to the section 5.1. ■

Lemma 2 *Let* $\mathcal{C} = \{j \in \mathbb{Z} \setminus \{0\} \mid |\lambda_j(T_\mathbf{W})| \geq c\}$ *and denote* $\mathcal{S}$ *the subspace spanned by the eigenfunctions* $\{\varphi_j(T_\mathbf{W})\}_{j \notin \mathcal{C}}$. *Then,* $\varphi_j(T_{\mathbf{W}_n}) \to \varphi_j(T_\mathbf{W})$ *weakly for* $j \in \mathcal{C}$ *and* $\varphi_j(T_{\mathbf{W}_n}) \to \Psi \in \mathcal{S}$ *for* $j \notin \mathcal{C}$, *where* Ψ *completes* $\varphi_j(T_\mathbf{W})$ *as an orthonormal basis.*

Proof: Refer to the section 5.2 ■

2 Convergence of the GFT to the WFT

When a sequence of graph signals converges to a bandlimited graphon signal, we can show that the GFT converges to the WFT as long as the limit graphon is *non-derogatory* (Def. 1). This is stated and proved in Thm. 1.

Theorem 1 (Convergence of GFT to WFT) *Let $\{(\mathbf{G}_n, \mathbf{x}_n)\}$ be a sequence of graph signals converging to the c-bandlimited graphon signal $(\mathbf{W}, X)$ in the sense of Def. 2, where $\mathbf{W}$ is non-derogatory. Then, there exists a sequence of permutations $\{\pi_n\}$ such that*

$$\begin{gathered} GFT\{(\pi_n(\mathbf{G}_n), \pi_n(\mathbf{x}_n))\} \to WFT\{(\mathbf{W}, X)\} \\ \text{and} \quad iGFT\{\hat{\mathbf{x}}_n\} \to iWFT\{\hat{X}\} \end{gathered}$$

where $\hat{\mathbf{x}}_n$ is the GFT of $(\pi_n(\mathbf{G}_n), \pi_n(\mathbf{x}_n))$ and $\hat{X}$ the WFT of $(\mathbf{W}, X)$. The eigenvalues of $\mathbf{W}$ and of the $\pi_n(\mathbf{G}_n)$ are assumed ordered with indices $j \in \mathbb{Z} \setminus \{0\}$ according to their sign and in decreasing order of absolute value.

Thm. 1 relates the GFT, a "discrete" transform under the probabilistic interpretation of graphons, to the WFT, a "continuous" Fourier transform for graphon signals. This makes for an interesting parallel with the relationship between the discrete Fourier transform (DFT) and the Fourier series for continuous time signals. It also allows drawing conclusions about the spectra of immeasurable or corrupted graph signals through analysis of the spectrum of the generating graphon signal when the latter is known. This is a consequence of both Thm. 1 and the fact that sampled sequences of graph signals converge to the generating graphon signal in probability.

3 Proof of convergence of the GFT to the WFT

Proof: [Proof of Thm. 1] We now prove that, since the finite set $\mathcal{L}$ converges to $\mathbb{Z} \setminus \{0\}$ as n goes to infinity, WFT$\{(\mathbf{W}_{\pi_n(\mathbf{G}_n)}, \pi_n(X_{\mathbf{G}_n}))\} \to$ WFT$\{(\mathbf{W}, X)\}$. We leave the dependence on $\pi_n(G_n)$ implicit and write $\mathbf{W}_n = \mathbf{W}_{\pi_n(\mathbf{G}_n)}$ and $X_n = \pi_n(X_{\mathbf{G}_n})$. Next, we use the eigenvector convergence result from the following lemma. Thm. 1 then follows from the fact that inner products are continuous in the product topology that they induce.

Starting with the eigenvectors with indices in $\mathcal{C}$, for any $\epsilon > 0$ it holds from Lemma 2 and from the convergence of X_n in L^2 that there exist n_1 and n_2 such that

$$\|\varphi_j(T_{\mathbf{W}_n}) - \varphi_j(T_{\mathbf{W}})\| \leq \frac{\epsilon}{2\|X\|}, \text{ for all } n > n_1$$
$$\text{and} \quad \|X_n - X\| \leq \frac{\epsilon}{2}, \text{ for all } n > n_2 \ .$$

Recall that $\|\varphi_j(T_{\mathbf{W}_n})\| \leq 1$ for all n and $j \in \mathcal{C}$ because the graphon spectral basis is orthonormal. Since the sequence $\{X_n\}$ is convergent, it is bounded and $\|X\| < \infty$. Let $m = \max\{n_1, n_2\}$. Then, it holds that

$$\begin{aligned}
|[\hat{X}_n]_j - [\hat{X}]_j| &= |\langle X_n, \varphi_j(T_{\mathbf{W}_n})\rangle - \langle X, \varphi_j(T_{\mathbf{W}})\rangle| \\
&= |\langle X_n - X, \varphi_j(T_{\mathbf{W}_n})\rangle + \langle X, \varphi_j(T_{\mathbf{W}_n}) - \varphi_j(T_{\mathbf{W}})\rangle| \\
&\leq \|X_n - X\|\|\varphi_j(T_{\mathbf{W}_n})\| + \|X\|\|\varphi_j(T_{\mathbf{W}_n}) - \varphi_j(T_{\mathbf{W}})\| \\
&\leq \frac{\epsilon}{2}\|\varphi_j(T_{\mathbf{W}_n})\| + \|X\|\frac{\epsilon}{2\|X\|} \leq \epsilon \text{ for all } n > m.
\end{aligned}$$

For $j \notin \mathcal{C}$, the eigenfunctions $\varphi_j(T_{\mathbf{W}_n})$ may not converge to $\varphi_j(T_{\mathbf{W}})$, but they do converge to some function $\Psi \in \mathcal{S}$. Given that the graphon signal $(\mathbf{W}, X)$ is bandlimited with bandwith c, we have $\langle X, \varphi_j(T_{\mathbf{W}})\rangle = 0$ for $j \notin \mathcal{C}$, so that X must be orthogonal to all functions in $\mathcal{S}$. Using the same argument as for $j \in \mathcal{C}$ yields that the remaining GFT coefficients also converge to the WFT. Formally,

$$\langle \varphi_j(T_{\mathbf{W}_n}), X_n\rangle \to \langle \Psi, X\rangle = 0 = \langle \varphi_j(T_{\mathbf{W}}), X\rangle \ .$$

Convergence of the iGFT to the iWFT follows directly from these results and from Lemma 2. Explicitly, use the triangle inequality to write

$$\begin{aligned}
&\left\| \sum_{j \in \mathbb{Z}\setminus\{0\}} [\hat{X}]_j \varphi_j(T_{\mathbf{W}}) - \sum_{j \in \mathbb{Z}\setminus\{0\}} [\hat{X}_n]_j \varphi_j(T_{\mathbf{W}_n}) \right\| \\
&\qquad \leq \sum_{j \in \mathbb{Z}\setminus\{0\}} \|[\hat{X}]_j \varphi_j(T_{\mathbf{W}}) - [\hat{X}]_j \varphi_j(T_{\mathbf{W}_n})\| \\
&\qquad + \sum_{j \in \mathbb{Z}\setminus\{0\}} \|[\hat{X}]_j \varphi_j(T_{\mathbf{W}_n}) - [\hat{X}_n]_j \varphi_j(T_{\mathbf{W}_n})\| \ .
\end{aligned}$$

Applying the Cauchy-Schwarz inequality and splitting the sums between $j \in \mathcal{C}$ and $j \notin \mathcal{C}$, we get

$$\begin{aligned} \left\| \sum_{j \in \mathbb{Z} \setminus \{0\}} [\hat{X}]_j \varphi_j(T_{\mathbf{W}}) - \sum_{j \in \mathbb{Z} \setminus \{0\}} [\hat{X}_n]_j \varphi_j(T_{\mathbf{W}_n}) \right\| \\ \leq \sum_{j \in \mathcal{C}} |[\hat{X}]_j| \|\varphi_j(T_{\mathbf{W}}) - \varphi_j(T_{\mathbf{W}_n})\| \\ + \sum_{j \in \mathcal{C}} |[\hat{X}]_j - [\hat{X}_n]_j| \|\varphi_j(T_{\mathbf{W}_n})\| \\ + \sum_{j \notin \mathcal{C}} |[\hat{X}_n]_j| \|\varphi_j(T_{\mathbf{W}_n})\| \to 0 \; . \end{aligned} \tag{1}$$

■

4 Important Remarks

We point out that the requirement that the graphon be non-derogatory is not very restrictive: as stated in the following proposition, the space of non-derogatory graphons is dense in the space of graphons.

Proposition 1 (Density of $\mathfrak{W}$) *Let $\mathfrak{W}$ denote the space of all bounded symmetric measurable functions $\mathbf{W} : [0,1]^2 \to \mathbb{R}$, i.e., the space of graphons. The space of* non-derogatory *graphons is dense in $\mathfrak{W}$.*

Proof: Refer to [1] ■

Prop. 1 tells us that, even if a graphon is derogatory, there exists a non-derogatory graphon arbitrarily close to it for which the GFT convergence result from Thm. 1 holds.

5 Proofs of preliminary results and lemmas

5.1 Proof of Lemma 1

Proof: The proof follows by direct computation. For $j \in \mathcal{L}$,

$$
\begin{aligned}
(T_{\mathbf{W_G}}\varphi_j)(u) &= \int_0^1 \mathbf{W_G}(u,v)\varphi_j(v)dv \\
&= \sqrt{n}\mathbb{I}\left(u \in I_k\right) \int_0^1 [\mathbf{S}]_{k\ell}[\mathbf{v}_j]_k \times \mathbb{I}\left(v \in I_\ell\right) dv \\
&= \sqrt{n}\mathbb{I}\left(u \in I_k\right) \sum_{\ell=1}^{n} [\mathbf{S}]_{k\ell}[\mathbf{v}_j]_k \int_{I_\ell} dv = \frac{[\mathbf{S}\mathbf{v}_j]_k}{n} \times \sqrt{n}\mathbb{I}\left(u \in I_k\right) \\
&= \frac{\lambda_j(\mathbf{S})}{n}\left[[\mathbf{v}_j]_k \times \sqrt{n}\mathbb{I}\left(u \in I_k\right)\right] = \lambda_j(T_{\mathbf{W_G}})\varphi_j(u).
\end{aligned}
$$

If $j \notin \mathcal{L}$, then $\langle \varphi_j, \varphi_k \rangle = 0$ for all $k \in \mathcal{L}$. In this case, we can trivially write $(T_{\mathbf{W_G}}\varphi_j)(u) = 0 = \lambda_j(T_{\mathbf{W_G}})\varphi_j(u)$. Note that since the $\mathbf{v}_k$ are orthonormal, so are the $\{\varphi_k(T_{\mathbf{W_G}})\}$ and therefore a basis completion $\{\varphi_j\}$ can always be obtained. To conclude, compute for $j \in \mathcal{L}$

$$
\begin{aligned}
[\hat{\phi}_{\mathbf{G}}]_j &= \int_0^1 \varphi_j(v)\phi_{\mathbf{G}}(v)dv \\
&= \sqrt{n}\int_0^1 [\mathbf{v}_j]_\ell[\mathbf{x}]_\ell \times \mathbb{I}\left(v \in I_\ell\right) dv \\
&= \sqrt{n}\sum_{\ell=1}^{n}[\mathbf{v}_j]_\ell[\mathbf{x}]_\ell \int_{I_\ell} dv = \frac{\mathbf{v}_j^{\mathsf{T}}\mathbf{x}}{\sqrt{n}} = \frac{[\hat{\mathbf{x}}]_j}{\sqrt{n}}.
\end{aligned}
$$

If $j \notin \mathcal{L}$, recall that since the $\{\mathbf{v}_j\}$ form a basis of $\mathbb{R}^n$, we can write $\mathbf{x} = \sum_{k\in\mathcal{L}} c_k\mathbf{v}_k$. Hence,

$$
\begin{aligned}
[\hat{\phi}_{\mathbf{G}}]_j &= \int_0^1 \varphi_j(v)\phi_{\mathbf{G}}(v)dv \\
&= \int_0^1 [\mathbf{x}]_\ell \times \mathbb{I}\left(v \in I_\ell\right) \varphi_j(v)dv \\
&= \int_0^1 \sum_{k\in\mathcal{L}} c_k[\mathbf{v}_k]_\ell \times \mathbb{I}\left(v \in I_\ell\right) \varphi_j(v)dv \\
&= \frac{1}{\sqrt{n}}\sum_{k\in\mathcal{L}} c_k \int_0^1 \varphi_k(v)\varphi_j(v)dv = 0.
\end{aligned}
$$

■

5.2 Proof of lemma 2

To prove Lemma 2, we require the following lema.

Lemma 3 (Eigenvalue convergence) *Let $\{\mathbf{G}_n\}$ be a sequence of graphs with eigenvalues $\{\lambda_j(\mathbf{S}_n)\}_{j\in\mathbb{Z}\setminus\{0\}}$, and $\mathbf{W}$ a graphon with eigenvalues $\{\lambda_j(T_{\mathbf{W}})\}_{j\in\mathbb{Z}\setminus\{0\}}$. Assume that, in both cases, the eigenvalues are ordered by decreasing order of absolute value and indexed according to their sign. If $\{\mathbf{G}_n\}$ converges to $\mathbf{W}$, then, for all j*

$$\lim_{n\to\infty}\frac{\lambda_j(\mathbf{S}_n)}{n} = \lim_{n\to\infty}\lambda_j(T_{\mathbf{W}_{\mathbf{G}_n}}) = \lambda_j(T_{\mathbf{W}})\ . \tag{2}$$

Proof:

Recall that since the sequence $\{\mathbf{G}_n\}$ converges to $\mathbf{W}$, the density of homomorphisms for any finite graph also converges. The result then follows by choosing a homomorphism connected to the eigenvalues of their induced operators, namely the k-cycle $\mathbf{C}_k$. Indeed, notice that for any graphon $\mathbf{W}'$ and $k \geq 2$, we have, by definition, that $t(\mathbf{C}_k, \mathbf{W}') = \sum_{i\in\mathbb{Z}\setminus\{0\}}\lambda_i(T_{\mathbf{W}'})^k$. Hence,

$$\lim_{n\to\infty}\sum_{i\in\mathbb{Z}\setminus\{0\}}\lambda_i(T_{\mathbf{W}_n})^k = \sum_{i\in\mathbb{Z}\setminus\{0\}}\lambda_i(T_{\mathbf{W}})^k, \text{ for } k\geq 2 \tag{3}$$

where $T_{\mathbf{W}_n} = T_{\mathbf{W}_{\mathbf{G}_n}}$. It now suffices to show that (3) implies $\lambda_i(T_{\mathbf{W}_n}) \to \lambda_i(T_{\mathbf{W}})$.

We start by bounding the eigenvalues of any graphon $\mathbf{W}'$ in terms of its density of homomorphisms. In particular, for $k = 4$ we obtain that

$$\begin{aligned}
\sum_{i=1}^{m}\lambda_i(T_{\mathbf{W}'})^4 \leq \sum_{i\in\mathbb{Z}\setminus\{0\}}\lambda_i(T_{\mathbf{W}'})^4 = t(\mathbf{C}_4,\mathbf{W}') \Rightarrow \\
\lambda_m(T_{\mathbf{W}'}) \leq \left[\frac{t(\mathbf{C}_4,\mathbf{W}')}{m}\right]^{1/4} \text{ and} \\
\sum_{i=-m}^{-1}\lambda_i(T_{\mathbf{W}'})^4 \leq \sum_{i\in\mathbb{Z}\setminus\{0\}}\lambda_i(T_{\mathbf{W}'})^4 = t(\mathbf{C}_4,\mathbf{W}') \Rightarrow \\
\lambda_{-m}(T_{\mathbf{W}'}) \geq -\left[\frac{t(\mathbf{C}_4,\mathbf{W}')}{m}\right]^{1/4}.
\end{aligned}$$

Since $t(\mathbf{C}_4, \mathbf{W}_n)$ is a convergent sequence, it has a bound B, which implies that

$$|\lambda_i(T_{\mathbf{W}_n})| \leq \left(\frac{B}{|i|}\right)^{1/4}, \text{ for all } i \in \mathbb{Z} \setminus \{0\}. \tag{4}$$

Note that for $k \geq 5$, we can take the limit in (3) term-by-term since, as $|\lambda_i(T_{\mathbf{W}_n})^k| \leq (B/|i|)^{k/4}$ and the series $\sum_i (B/|i|)^{k/4}$ is convergent for $k > 4$, $\sum_{i \in \mathbb{Z} \setminus \{0\}} |\lambda_i(T_{\mathbf{W}_n})^k|$ also converges. Hence, from (3), we have

$$\lim_{n\to\infty} \sum_{i \in \mathbb{Z}\setminus\{0\}} \lambda_i(T_{\mathbf{W}_n})^k = \sum_{i \in \mathbb{Z}\setminus\{0\}} \zeta_i^k = \sum_{i \in \mathbb{Z}\setminus\{0\}} \lambda_i(T_{\mathbf{W}})^k \tag{5}$$

for $k \geq 5$, where $\zeta_i^k = \lim_{n\to\infty} \lambda_i(T_{\mathbf{W}_n})^k$.

To conclude, we proceed by induction over an ordering of the sequence of eigenvalues $\lambda_i(T_{\mathbf{W}})$, namely over i_ℓ, $\ell = 1, 2, \ldots$, such that $|\lambda_{i_1}(T_{\mathbf{W}})| \geq |\lambda_{i_2}(T_{\mathbf{W}})| \geq \cdots \geq |\lambda_{i_\ell}(T_{\mathbf{W}})|$. Suppose that $\zeta_{i_\ell} = \lambda_{i_\ell}(T_{\mathbf{W}})$ for $\ell < \ell^*$ and let $\lambda_{i_{\ell^*}}(T_{\mathbf{W}})$ be of multiplicity a and appear b times in the sequence $\{\zeta_i\}$ and $-\lambda_{i_{\ell^*}}(T_{\mathbf{W}})$ be of multiplicity a' and appear b' times in $\{\zeta_i\}$. The identity in (5) then reduces to

$$\left[b + (-1)^k b'\right] + \sum_{\ell > \ell^*} \left(\frac{\zeta_{i_\ell}}{\lambda_{i_{\ell^*}}(T_{\mathbf{W}})}\right)^k =$$

$$\left[a + (-1)^k a'\right] + \sum_{\ell > \ell^*} \left(\frac{\lambda_{i_\ell}(T_{\mathbf{W}})}{\lambda_{i_{\ell^*}}(T_{\mathbf{W}})}\right)^k, \text{ for } k \geq 5,$$

where we divided both sides by $\lambda_{i_{\ell^*}}(T_{\mathbf{W}})^k$. Due to the ordering of the λ_{i_ℓ}, for $k \to \infty$ through the even numbers we get $b + b' = a + a'$ and through the odd numbers we get $b - b' = a - a'$. Immediately, we have that $a = a'$ and $b = b'$, so that $\zeta_{i_{\ell^*}} = \lambda_{i_{\ell^*}}$. Although this argument assumes $\zeta_{i_\ell} < \lambda_{i_{\ell^*}}$ for all $\ell > \ell^*$, applying the same procedure to an ordering of the sequence $\{\zeta_i\}$ yields the same conclusion. ■

We will also require the following well known result about the perturbation of self-adjoint operators. For σ a subset of the eigenvalues of a self-adjoint operator T, define the spectral projection $E_T(\sigma)$ as the projection onto the subspace spanned by the eigenfunctions relative to those eigenvalues in σ. Then,

Proposition 2 *Let T and T' be two self-adjoint operators on a separable Hilbert space $\mathcal{H}$ whose spectra are partitioned as $\sigma \cup \Sigma$ and $\omega \cup \Omega$ respectively, with $\sigma \cap$*

$\Sigma = \emptyset$ and $\omega \cap \Omega = \emptyset$. If there exists $d > 0$ such that $\min_{x \in \sigma, y \in \Omega} |x - y| \geq d$ and $\min_{x \in \omega, y \in \Sigma} |x - y| \geq d$, then

$$\| E_T(\sigma) - E_{T'}(\omega) \| \leq \frac{\pi}{2} \frac{\| T - T' \|}{d} \tag{6}$$

Proof: See [2]. ∎

Lastly, we need two results related to the graphon norm. The first, presented in Lemma 3, states that if a sequence of graphs converges to a graphon in the homomorphism density sense, it also converges in the cut norm. The cut norm of a graphon $\mathbf{W} : [0,1]^2 \to [0,1]$ is defined as [3, eq. (8.13)]

$$\|\mathbf{W}\|_\square = \sup_{S,T \subseteq [0,1]} \left| \int_{S \times T} \mathbf{W}(u,v) du dv \right|.$$

The second, here presented as Prop. 3, is due to [3, Thm. 11.57] and bounds the L^2-induced norm of the graphon operator by is cut norm.

Lemma 3 (Cut norm convergence) *If $\{\mathbf{G}_n\} \to \mathbf{W}$ in the homomorphism density sense, then the there exists a sequence of permutations $\{\pi_n\}$ such that*

$$\|\mathbf{W}_{\pi_n(\mathbf{G}_n)} - \mathbf{W}\|_\square \to 0$$

where $\mathbf{W}_{\mathbf{G}_n}$ is the graphon induced by the graph $\mathbf{G}_n$.

Proof: See [3, Thm. 11.57]. ∎

Proposition 3 *Let $T_\mathbf{W}$ be the operator induced by the graphon $\mathbf{W}$. Then, $\|\mathbf{W}\|_\square \leq \| T_\mathbf{W} \| \leq \sqrt{8 \|\mathbf{W}\|_\square}$.*

This is a direct consequence of [4, Thm. 3.7(a)] and of the fact that $t(\mathbf{C}_2, \mathbf{W})$ is the Hilbert-Schmidt norm of $T_\mathbf{W}$, which dominates the L^2-induced operator norm.

We can now proceed with the proof of our lemma:

Proof: [Proof of Lemma 2] For $j \in \mathcal{C}$, let $\sigma = \lambda_j(T_{\mathbf{W}})$, $\Sigma = \{\lambda_i(T_{\mathbf{W}})\}_{i \neq j}$, $\omega = \lambda_j(T_{\mathbf{W}_n})$, and $\Omega = \{\lambda_i(T_{\mathbf{W}_n})\}_{i \neq j}$ in Prop. 2 to get

$$|||E_j - E_{jn}||| \leq \frac{\pi}{2} \frac{|||T_{\mathbf{W}_n} - T_{\mathbf{W}}|||}{d_{jn}} \tag{7}$$

where E_j and E_{jn} are the spectral projections of $T_{\mathbf{W}}$ and $T_{\mathbf{W}_n}$ with respect to their j-th eigenvalue and

$$d_{jn} = \min \big(|\lambda_j - \lambda_{j+1}(T_{\mathbf{W}_n})|, |\lambda_j - \lambda_{j-1}(T_{\mathbf{W}_n})|, \\ |\lambda_{j+1} - \lambda_j(T_{\mathbf{W}_n})|, |\lambda_{j-1} - \lambda_j(T_{\mathbf{W}_n})| \big),$$

where we omitted the dependence on $\mathbf{W}$ by writing $\lambda_j = \lambda_j(T_{\mathbf{W}})$.

Fix $\epsilon > 0$. From Lemma 3, we know we can find n_1 such that $|d_{jn} - \delta_j| \leq \delta_j/2$ for all $n > n_1$, where

$$\delta_j = \min \left(|\lambda_j - \lambda_{j+1}|, |\lambda_j - \lambda_{j-1}| \right) .$$

Since $\mathbf{W}$ is non-derogatory, $\delta_j > 0$. Additionally, the cut norm convergence of graphon sequences (Lemma 3) together with Prop. 3 implies there exists n_2 such that $|||T_{\mathbf{W}_n} - T_{\mathbf{W}}||| \leq \epsilon\delta_j/\pi$. Hence, for all $n > \max(n_1, n_2)$ it holds from (7) that

$$|||E_j - E_{jn}||| \leq \frac{\pi}{2} \frac{\epsilon\delta_j/\pi}{\delta_j/2} = \epsilon. \tag{8}$$

Since ϵ is arbitrary, (8) proves that the projections onto the eigenfunctions of the same eigenvalue converge. I.e., the eigenfunction sequence $\varphi_j(T_{\mathbf{W}_n})$ itself converges weakly.

To proceed, let us apply Prop. 2 to the subspace spanned by the remaining eigenfunctions with indices not in $\mathcal{C}$. Let $\sigma = \{\lambda_i(T_{\mathbf{W}})\}_{i \notin \mathcal{C}}$, $\Sigma = \{\lambda_i(T_{\mathbf{W}})\}_{i \in \mathcal{C}}$, $\omega = \{\lambda_i(T_{\mathbf{W}_n})\}_{i \notin \mathcal{C}}$, and $\Omega = \{\lambda_i(T_{\mathbf{W}_n})\}_{i \in \mathcal{C}}$ in (6) to get

$$|||E' - E'_n||| \leq \frac{\pi}{2} \frac{|||T_{\mathbf{W}_n} - T_{\mathbf{W}}|||}{d_n}, \tag{9}$$

where E' and E'_n are the projections onto the subspaces given by $\mathcal{S} = \text{span}\left(\{\varphi_i(T_{\mathbf{W}})\}_{i \notin \mathcal{C}}\right)$ and $\mathcal{S}_n = \text{span}\left(\{\varphi_i(T_{\mathbf{W}_n})\}_{i \notin \mathcal{C}}\right)$ respectively. From Prop. 2, the denominator d_n must satisfy $d_n \leq \min_{i \notin \mathcal{C}, i-\text{sgn}(i) \in \mathcal{C}} |\lambda_i(T_{\mathbf{W}_n}) - \lambda_{i-\text{sgn}(i)}(T_{\mathbf{W}})| = d^{(1)}$ and $d \leq \min_{i \notin \mathcal{C}, i-\text{sgn}(i) \in \mathcal{C}} |\lambda_i(T_{\mathbf{W}}) - \lambda_{i-\text{sgn}(i)}(T_{\mathbf{W}_n})| =$

$d^{(2)}$. For $j \in \mathcal{C}$, we have $|\lambda_j(T_{\mathbf{W}})| \geq c$ and so $d^{(1)} \geq \min_{i \notin \mathcal{C}} c - |\lambda_i(T_{\mathbf{W}_n})|$. As for $d^{(2)}$, there exists n_0 such that $d^{(2)} \geq \min_{i \notin \mathcal{C}} c - |\lambda_i(T_{\mathbf{W}})|$ for $n > n_0$ because $\lambda_j(T_{\mathbf{W}_n}) \to \lambda_j(T_{\mathbf{W}})$ for all j from Lemma 3. Thus, for $n > n_0$ Prop. 2 holds with d_n given by

$$d_n \leq \min\{\min_{i \notin \mathcal{C}} c - |\lambda_i(T_{\mathbf{W}_n})|, \\ \min_{i \notin \mathcal{C}} c - |\lambda_i(T_{\mathbf{W}})|\}$$

which is satisfied by $d_n = \inf_{i \notin \mathcal{C}} c - |\lambda_i(T_{\mathbf{W}_n})|$. Since the graphon $\mathbf{W}$ is non-derogatory, there exists an n_1 such that $d_n > 0$ for all $n > \max(n_0, n_1)$ and we can use the same argument as above to obtain that $E'_n \to E'$ in operator norm.

To see how this implies that $\varphi_i(T_{\mathbf{W}_n}) \to \Psi \in \mathcal{S}$ for all $i \notin \mathcal{C}$, suppose this is not the case. Then, $\|\Psi - E'(\Psi)\| \geq \epsilon > 0$ since $\Psi \notin \mathcal{S}$. Without loss of generality, we assume that $\|\Psi\| = 1$ (if not, simply normalize Ψ: since $\mathcal{S}$ is a subspace $\Psi \notin \mathcal{S} \Leftrightarrow K\Psi \notin \mathcal{S}$ for any $K > 0$). Notice, however, that there exists n' such that $\|\varphi_i(T_{\mathbf{W}_n}) - \Psi\| \leq \epsilon/8$ and $\|E'(\Psi) - E'_n(\Psi)\| \leq \epsilon/4$ for all $n > n'$, which implies that $\|\Psi - E'(\Psi)\| \leq \epsilon/2$, contradicting the hypothesis. Indeed,

$$\begin{aligned}\|\Psi - E'(\Psi)\| =& \|\Psi - \varphi_i(T_{\mathbf{W}_n}) + E'_n(\Psi) - E'(\Psi) + \\ & E'_n(\varphi_i(T_{\mathbf{W}_n}) - \Psi)\| \leq \|\Psi - \varphi_i(T_{\mathbf{W}_n})\| + \\ & \|E'_n(\Psi) - E'(\Psi)\| + \|E'_n(\varphi_i(T_{\mathbf{W}_n}) - \Psi)\|.\end{aligned}$$

Then, using Cauchy-Schwarz and the fact that E'_n is an orthogonal projection, i.e., $\|E'_n\| = 1$, yields

$$\|\Psi - E'(\Psi)\| \leq 2\|\Psi - \varphi_i(T_{\mathbf{W}_n})\| + \|E'_n(\Psi) - E'(\Psi)\|.$$

which for all $n > n'$ reduces to

$$\|\Psi - E'(\Psi)\| \leq \frac{\epsilon}{2} \tag{10}$$

contradicting the fact that $\Psi \notin \mathcal{S}$. ∎

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