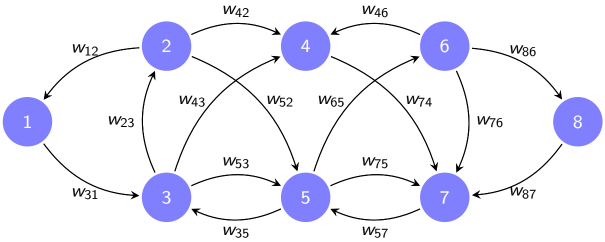
Graphs

- A graph is a triplet $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, which includes vertices $\mathcal{V}$, edges $\mathcal{E}$, and weights $\mathcal{W}$

⇒ Vertices or nodes are a set of n labels. Typical labels are $\mathcal{V} = \{1, \ldots, n\}$

⇒ Edges are ordered pairs of labels (i, j). We interpret $(i, j) \in \mathcal{E}$ as "i can be influenced by j."

⇒ Weights $w_{ij} \in \mathbb{R}$ are numbers associated to edges (i, j). "Strength of the influence of j on i."

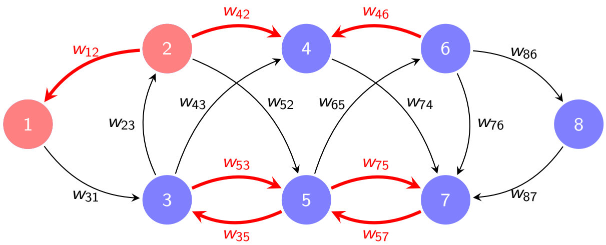


- Edge (i, j) is represented by an arrow pointing from j into i. Influence of node j on node i

  $\Rightarrow$ This is the opposite of the standard notation used in graph theory

- Edge (i, j) is different from edge (j, i) $\Rightarrow$ It is possible to have $(i, j) \in \mathcal{E}$ and $(j, i) \notin \mathcal{E}$

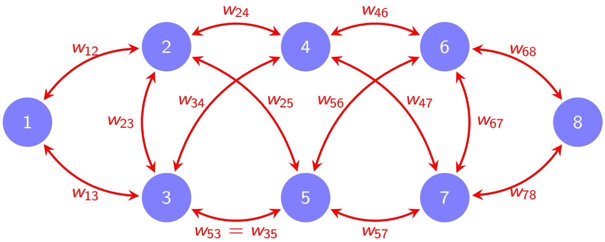
- If both edges are in the edge set, the weights can be different $\Rightarrow$ It is possible to have $w_{ij} \neq w_{ji}$



- A graph is symmetric or undirected if both, the edge set and the weight are symmetric

$\Rightarrow$ Edges come in pairs $\Rightarrow$ We have $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$

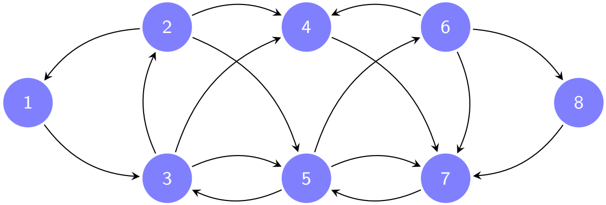
$\Rightarrow$ Weights are symmetric $\Rightarrow$ We must have $w_{ij} = w_{ji}$ for all $(i, j) \in \mathcal{E}$



- A graph is unweighted if it doesn't have weights

  $\Rightarrow$ Equivalently, we can say that all weights are units $\Rightarrow w_{ij} = 1$ for all $(i, j) \in \mathcal{E}$

- Unweighted graphs could be directed or symmetric



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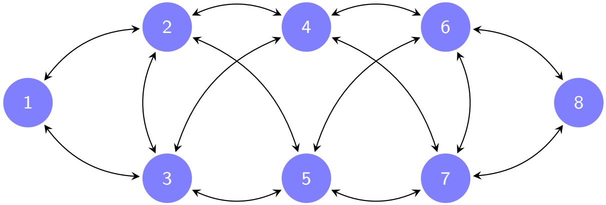
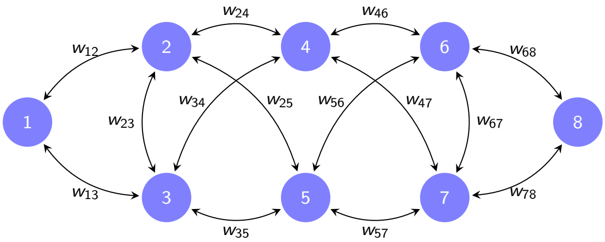


Weighted Symmetric Graphs

- Graphs can be directed or symmetric. Separately, they can be weighted or unweighted.

- Most of the graphs we encounter in practical situations are symmetric and weighted



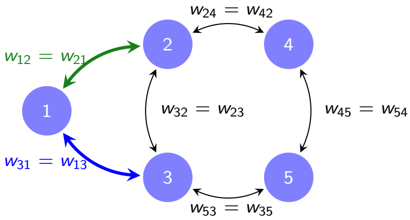
Graph Shift Operators

- Graphs have matrix representations. Which in this course, we call graph shift operators (GSOs)

- The adjacency matrix of graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ is the sparse matrix $\mathbf{A}$ with nonzero entries

$$A_{ij} = w_{ij}, \text{ for all } (i, j) \in \mathcal{E}$$

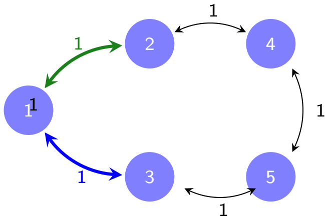
- If the graph is symmetric, the adjacency matrix is symmetric $\Rightarrow \mathbf{A} = \mathbf{A}^T$. As in the example



$$\mathbf{A} = \begin{bmatrix} 0 & w_{12} & w_{13} & 0 & 0 \\ w_{21} & 0 & w_{23} & w_{24} & 0 \\ w_{31} & w_{32} & 0 & 0 & w_{35} \\ 0 & w_{42} & 0 & 0 & w_{45} \\ 0 & 0 & w_{53} & w_{54} & 0 \end{bmatrix}.$$

- For the particular case in which the graph is unweighted. Weights interpreted as units

$$A_{ij} = 1, \qquad \text{for all} \quad (i,j) \in \mathcal{E}$$

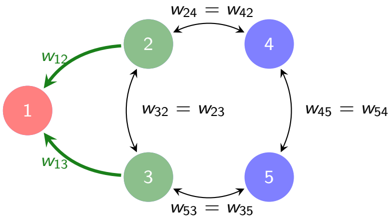


$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Neighborhoods and Degrees

- The neighborhood of node i is the set of nodes that influence i $\Rightarrow n(i) := \{j : (i,j) \in \mathcal{E}\}$

- Degree d_i of node i is the sum of the weights of its incident edges $\Rightarrow d_i = \sum_{j \in n(i)} w_{ij} = \sum_{j:(i,j) \in \mathcal{E}\}} w_{ij}$

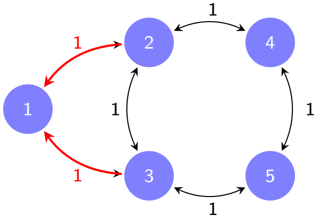


- Node 1 neighborhood $\Rightarrow n(1) = \{2, 3\}$

- Node 1 degree $\Rightarrow n(1) = w_{12} + w_{13}$

- The degree matrix is a diagonal matrix $\mathbf{D}$ with degrees as diagonal entries $\Rightarrow D_{ii} = d_i$

- Write in terms of adjacency matrix as $\mathbf{D} = \text{diag}(\mathbf{A1})$. Because $(\mathbf{A1})_i = \sum_j w_{ij} = d_i$



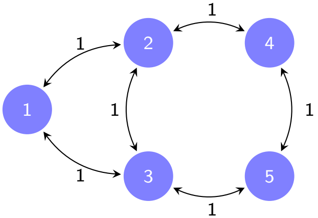
$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

- The Laplacian matrix of a graph with adjacency matrix $\mathbf{A}$ is $\Rightarrow \mathbf{L} = \mathbf{D} - \mathbf{A} = \text{diag}(\mathbf{A1}) - \mathbf{A}$
- Can also be written explicitly in terms of graph weights $A_{ij} = w_{ij}$

$\Rightarrow$ Off diagonal entries $\Rightarrow L_{ij} = -A_{ij} = -w_{ij}$

$\Rightarrow$ Diagonal entries $\Rightarrow L_{ii} = d_i = \sum_{j \in n(i)} w_{ij}$

$$\mathbf{L} = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 3 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$



- Normalized adjacency and Laplacian matrices express weights relative to the nodes' degrees

- Normalized adjacency matrix $\Rightarrow \bar{\mathbf{A}} := \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$ $\Rightarrow$ Results in entries $(\bar{\mathbf{A}})_{ij} = \dfrac{w_{ij}}{\sqrt{d_i d_j}}$

- The normalized adjacency is symmetric if the graph is symmetric $\Rightarrow \bar{\mathbf{A}}^T = \bar{\mathbf{A}}$.

- Normalized Laplacian matrix $\Rightarrow \bar{\mathbf{L}} := \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$. Same normalization of adjacency matrix

- Given definitions normalized representations $\Rightarrow \bar{\mathbf{L}} \;=\; \mathbf{D}^{-1/2}\Big(\mathbf{D}-\mathbf{A}\Big)\mathbf{D}^{-1/2} \;=\; \mathbf{I}-\bar{\mathbf{A}}$

  $\Rightarrow$ The normalized Laplacian and adjacency are essentially the same linear transformation.

- Normalized operators are more homogeneous. The entries in the vector $\mathbf{A1}$ tend to be similar.

- The Graph Shift Operator $\mathbf{S}$ is a stand in for any of the matrix representations of the graph

Adjacency Matrix	Laplacian Matrix	Normalized Adjacency	Normalized Laplacian
$\mathbf{S} = \mathbf{A}$	$\mathbf{S} = \mathbf{L}$	$\mathbf{S} = \bar{\mathbf{A}}$	$\mathbf{S} = \bar{\mathbf{L}}$

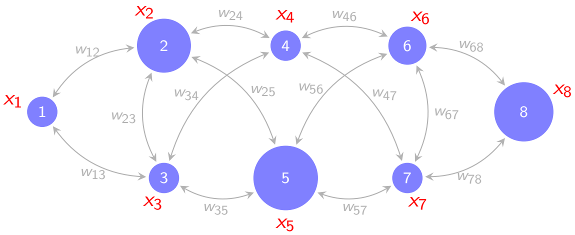
- If the graph is symmetric, the shift operator $\mathbf{S}$ is symmetric $\Rightarrow \mathbf{S} = \mathbf{S}^T$

- The specific choice matters in practice but most of results and analysis hold for any choice of $\mathbf{S}$

Graph Signals

- Graph Signals are supported on a graph. They are the objets we process in Graph Signal Processing

- Consider a given graph $\mathcal{G}$ with n nodes and shift operator $\mathbf{S}$
- A graph signal is a vector $\mathbf{x} \in \mathbb{R}^n$ in which component x_i is associated with node i
- To emphasize that the graph is intrinsic to the signal we may write the signal as a pair $\Rightarrow$ $(\mathbf{S}, \mathbf{x})$



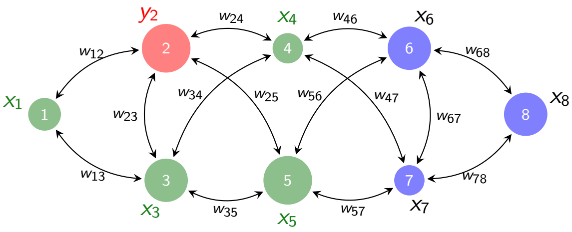
- The graph is an expectation of proximity or similarity between components of the signal $\mathbf{x}$

- Multiplication by the graph shift operator implements diffusion of the signal over the graph

- Define diffused signal $\mathbf{y} = \mathbf{S}\mathbf{x}$ $\Rightarrow$ Components are $y_i = \sum_{j \in n(i)} w_{ij}\, x_j \;=\; \sum_{j} w_{ij}\, x_j$

  $\Rightarrow$ Stronger weights contribute more to the diffusion output

  $\Rightarrow$ Codifies a local operation where components are mixed with components of neighboring nodes.

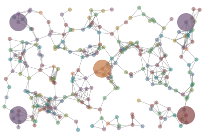


The Diffusion Sequence

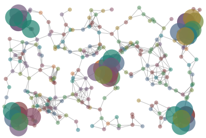
- Compose the diffusion operator to produce diffusion sequence $\Rightarrow$ defined recursively as

$$\mathbf{x}^{(k+1)} = \mathbf{S}\mathbf{x}^{(k)}, \qquad \text{with} \quad \mathbf{x}^{(0)} = \mathbf{x}$$

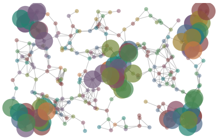
- Can unroll the recursion and write the diffusion sequence as the power sequence $\Rightarrow \mathbf{x}^{(k)} = \mathbf{S}^k\mathbf{x}$



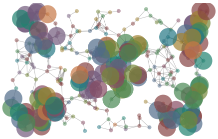
$\mathbf{x}^{(0)} = \mathbf{x} \;\; = \mathbf{S}^0\mathbf{x}$



$\mathbf{x}^{(1)} = \mathbf{S}\mathbf{x}^{(0)} \;\; = \mathbf{S}^1\mathbf{x}$



$\mathbf{x}^{(2)} = \mathbf{S}\mathbf{x}^{(1)} \;\; = \mathbf{S}^2\mathbf{x}$



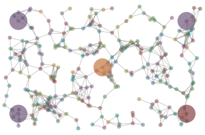
$\mathbf{x}^{(3)} = \mathbf{S}\mathbf{x}^{(2)} \;\; = \mathbf{S}^3\mathbf{x}$

- The kth element of the diffusion sequence $x^{(k)}$ diffuses information to k-hop neighborhoods

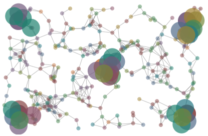
  $\Rightarrow$ One reason why we use the diffusion sequence to define graph convolutions

- We have two definitions. One recursive. The other one using powers of $\mathbf{S}$

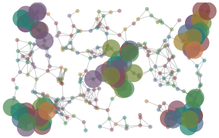
  $\Rightarrow$ Always implement the recursive version. The power version is good for analysis



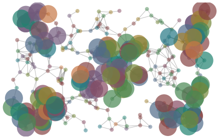
$\mathbf{x}^{(0)} = \mathbf{x} = \mathbf{S}^0\mathbf{x}$



$\mathbf{x}^{(1)} = \mathbf{S}\mathbf{x}^{(0)} = \mathbf{S}^1\mathbf{x}$



$\mathbf{x}^{(2)} = \mathbf{S}\mathbf{x}^{(1)} = \mathbf{S}^2\mathbf{x}$



$\mathbf{x}^{(3)} = \mathbf{S}\mathbf{x}^{(2)} = \mathbf{S}^3\mathbf{x}$

Graph Convolutional Filters

- Graph convolutional filters are the tool of choice for the linear processing of graph signals

- Given graph shift operator $\mathbf{S}$ and coefficients h_k, a graph filter is a polynomial (series) on $\mathbf{S}$

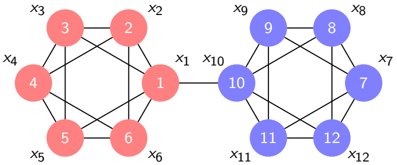
$$\mathbf{H}(\mathbf{S}) = \sum_{k=0}^{\infty} h_k \mathbf{S}^k$$

- The result of applying the filter $\mathbf{H}(\mathbf{S})$ to the signal $\mathbf{x}$ is the signal

$$\mathbf{y} \;=\; \mathbf{H}(\mathbf{S})\,\mathbf{x} \;=\; \sum_{k=0}^{\infty} h_k \mathbf{S}^k \mathbf{x}$$

- We say that $\mathbf{y} = \mathbf{h} \star_{\mathbf{S}} \mathbf{x}$ is the graph convolution of the filter $\mathbf{h} = \{h_k\}_{k=0}^{\infty}$ with the signal $\mathbf{x}$

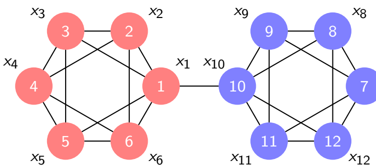
- Graph convolutions aggregate information growing from local to global neighborhoods
- Consider a signal $\mathbf{x}$ supported on a graph with shift operator $\mathbf{S}$. Along with filter $\mathbf{h} = \{h_k\}_{k=0}^{K-1}$



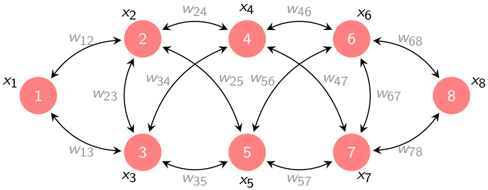
- Graph convolution output $\Rightarrow \mathbf{y} = \mathbf{h} \star_{\mathbf{S}} \mathbf{x} = \; h_0 \mathbf{S}^0 \mathbf{x} \; + h_1 \mathbf{S}^1 \mathbf{x} \; + h_2 \mathbf{S}^2 \mathbf{x} \; + h_3 \mathbf{S}^3 \mathbf{x} \; + \ldots = \sum_{k=0}^{K-1} h_k \mathbf{S}^k \mathbf{x}$

- The same filter $\mathbf{h} = \{h_k\}_{k=0}^{\infty}$ can be executed in multiple graphs $\Rightarrow$ We can transfer the filter

Graph Filter on a Graph



Same Graph Filter on Another Graph

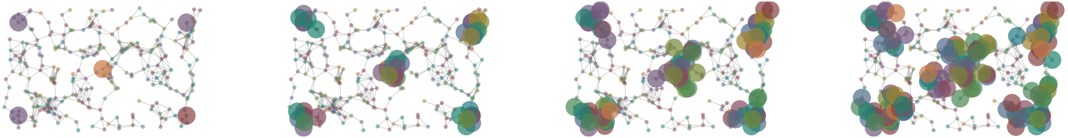


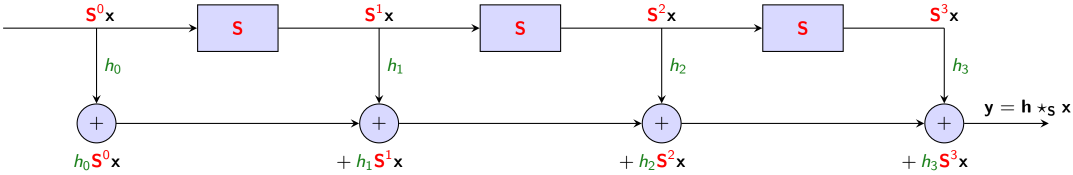
- Graph convolution output $\Rightarrow \mathbf{y} = \mathbf{h} \star_{\mathbf{S}} \mathbf{x} = h_0 \mathbf{S}^0 \mathbf{x} + h_1 \mathbf{S}^1 \mathbf{x} + h_2 \mathbf{S}^2 \mathbf{x} + h_3 \mathbf{S}^3 \mathbf{x} + \ldots = \sum_{k=0}^{\infty} h_k \mathbf{S}^k \mathbf{x}$

- Output depends on the filter coefficients $\mathbf{h}$, the graph shift operator $\mathbf{S}$ and the signal $\mathbf{x}$

Graph Convolutional Filters as Diffusion Operators

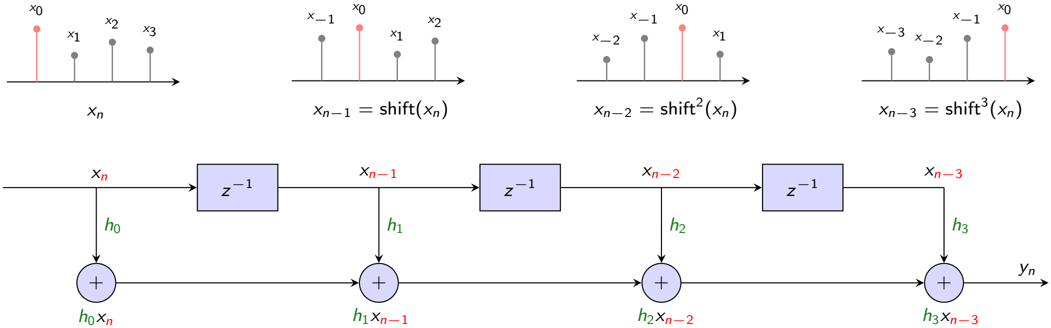
- A graph convolution is a weighted linear combination of the elements of the diffusion sequence
- Can represent graph convolutions with a shift register $\Rightarrow$ Convolution $\equiv$ Shift. Scale. Sum





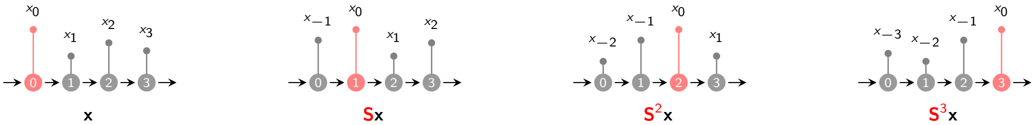
Time Convolutions as a Particular Case of Graph Convolutions

- Convolutional filters process signals in time by leveraging the time shift operator



- The time convolution is a linear combination of time shifted inputs $\Rightarrow y_n = \sum_{k=0}^{K-1} h_k x_{n-k}$

- Time signals are representable as graph signals supported on a line graph $\mathbf{S}$ $\Rightarrow$ The pair $(\mathbf{S}, \mathbf{x})$



- Time shift is reinterpreted as multiplication by the adjacency matrix $\mathbf{S}$ of the line graph

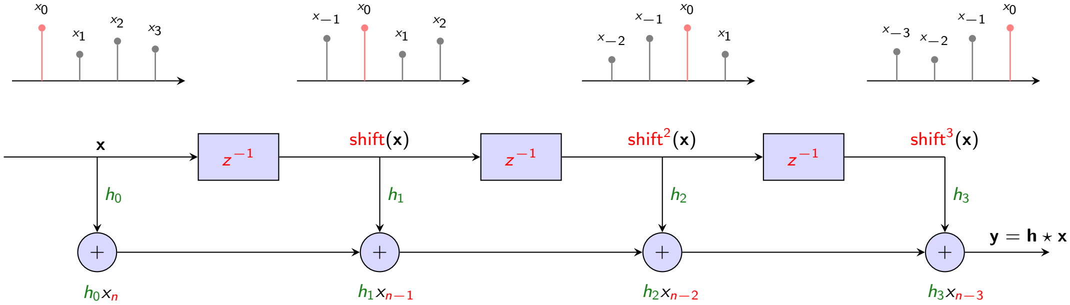
$$\mathbf{S}^3\mathbf{x} = \mathbf{S}\left[\mathbf{S}^2\mathbf{x}\right] = \mathbf{S}\left[\mathbf{S}\left(\mathbf{S}\mathbf{x}\right)\right] = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} \vdots \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ x_{-3} \\ x_{-2} \\ x_{-1} \\ x_0 \\ \vdots \end{bmatrix}$$

- Components of the shift sequence are powers of the adjacency matrix applied to the original signal

$\Rightarrow$ We can rewrite convolutional filters as polynomials on $\mathbf{S}$, the adjacency of the line graph

The Convolution as a Polynomial on the Line Adjacency

- The convolution operation is a linear combination of shifted versions of the input signal
- But we now know that time shifts are multiplications with the adjacency matrix $\mathbf{S}$ of line graph



- Time convolution is a polynomial on adjacency matrix of line graph $\Rightarrow \mathbf{y} = \mathbf{h} \star \mathbf{x} = \sum_{k=0}^{K-1} h_k \mathbf{S}^k \mathbf{x}$

The Convolution as a Polynomial on the Line Adjacency

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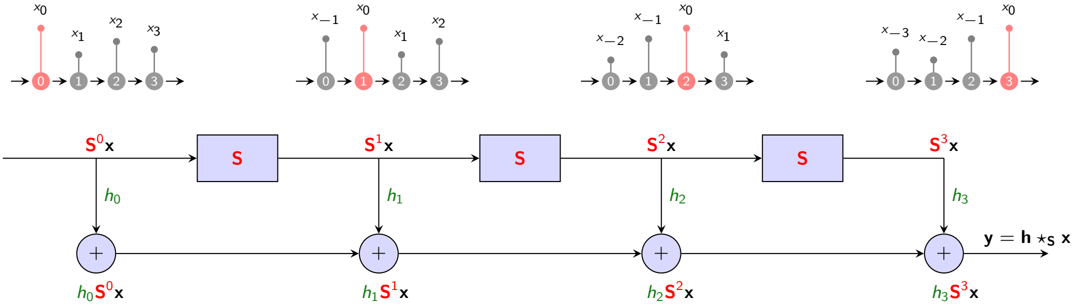
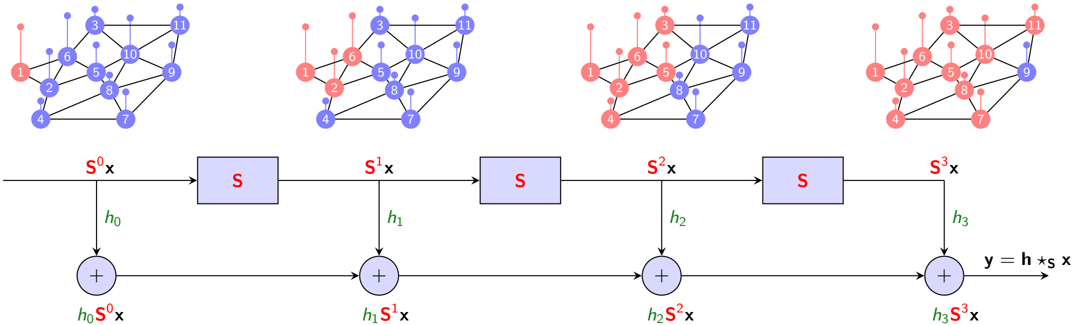
- But we now know that time shifts are multiplications with the adjacency matrix $\mathbf{S}$ of line graph



- Time convolution is a polynomial on adjacency matrix of line graph $\Rightarrow \mathbf{y} = \mathbf{h} \star \mathbf{x} = \sum_{k=0}^{K-1} h_k \mathbf{S}^k \mathbf{x}$

- If we let **S** be the shift operator of an arbitrary graph we recover the graph convolution



Graph Fourier Transform

- The Graph Fourier Transform (GFT) is a tool for analyzing graph information processing systems

Eigenvectors and Eigenvalues of Shift Operator

- We work with symmetric graph shift operators $\Rightarrow \mathbf{S} = \mathbf{S}^H$

- Introduce eigenvectors $\mathbf{v}_i$ and eigenvalues λ_i of graph shift operator $\mathbf{S}$ $\Rightarrow \mathbf{S}\mathbf{v}_i = \lambda_i \mathbf{v}_i$

  $\Rightarrow$ For symmetric $\mathbf{S}$ eigenvalues are real. We have ordered them $\Rightarrow \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_n$

- Define eigenvector matrix $\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ and eigenvalue matrix $\mathbf{\Lambda} = \text{diag}([\lambda_1; \ldots; \lambda_n])$

  $\Rightarrow$ Eigenvector decomposition of Graph Shift Operator $\Rightarrow \mathbf{S} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$. With $\mathbf{V}^H\mathbf{V} = \mathbf{I}$

Graph Fourier Transform

Given a graph shift operator $\mathbf{S} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$, the graph Fourier transform (GFT) of graph signal $\mathbf{x}$ is

$$\tilde{\mathbf{x}} = \mathbf{V}^H \mathbf{x}$$

- The GFT is a projection on the eigenspace of the graph shift operator.
- We say $\tilde{\mathbf{x}}$ is a graph frequency representation of $\mathbf{x}$. A representation in the graph frequency domain

Inverse Graph Fourier Transform

Given a graph shift operator $\mathbf{S} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$, the inverse graph Fourier transform (iGFT) of GFT $\tilde{\mathbf{x}}$ is

$$\tilde{\tilde{\mathbf{x}}} = \mathbf{V}\,\tilde{\mathbf{x}}$$

- Given that $\mathbf{V}^H\mathbf{V} = \mathbf{I}$, the iGFT of the GFT of signal $\mathbf{x}$ recovers the signal $\mathbf{x}$

$$\tilde{\tilde{\mathbf{x}}} = \mathbf{V}\,\tilde{\mathbf{x}} = \mathbf{V}\left(\mathbf{V}^H\mathbf{x}\right) = \mathbf{I}\mathbf{x} = \mathbf{x}$$

Graph Frequency Response of Graph Filters

- Graph filters admit a pointwise representation when projected into the shift operator's eigenspace

Theorem (Graph frequency representation of graph filters)

Consider graph filter $\mathbf{h}$ with coefficients h_k, graph signal $\mathbf{x}$ and the filtered signal $\mathbf{y} = \sum_{k=0}^{\infty} h_k \mathbf{S}^k \mathbf{x}$.

The GFTs $\tilde{\mathbf{x}} = \mathbf{V}^H \mathbf{x}$ and $\tilde{\mathbf{y}} = \mathbf{V}^H \mathbf{y}$ are related by

$$\tilde{\mathbf{y}} = \sum_{k=0}^{\infty} h_k \mathbf{\Lambda}^k \tilde{\mathbf{x}}$$

- The same polynomial but on different variables. One on $\mathbf{S}$. The other on eigenvalue matrix $\mathbf{\Lambda}$

Proof: Since $\mathbf{S} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^H$, can write shift operator powers as $\mathbf{S}^k = \mathbf{V}\boldsymbol{\Lambda}^k\mathbf{V}^H$. Therefore filter output is

$$\mathbf{y} \;=\; \sum_{k=0}^{\infty} h_k \mathbf{S}^k \mathbf{x} \;=\; \sum_{k=0}^{\infty} h_k \mathbf{V}\boldsymbol{\Lambda}^k\mathbf{V}^H \mathbf{x}$$

- Multiply both sides by $\mathbf{V}^H$ on the left $\Rightarrow \mathbf{V}^H\mathbf{y} \;=\; \mathbf{V}^H \sum_{k=0}^{\infty} h_k \mathbf{V}\boldsymbol{\Lambda}^k\mathbf{V}^H \mathbf{x}$

- Copy and identify terms. Output GFT $\mathbf{V}^H\mathbf{y} = \tilde{\mathbf{y}}$. Input GFT $\mathbf{V}^H\mathbf{x} = \tilde{\mathbf{x}}$. Cancel out $\mathbf{V}^H\mathbf{V}$

$$\mathbf{V}^H\mathbf{y} = \mathbf{V}^H \sum_{k=0}^{\infty} h_k \mathbf{V}\boldsymbol{\Lambda}^k\mathbf{V}^H \mathbf{x} \qquad \Rightarrow \qquad \tilde{\mathbf{y}} = \sum_{k=0}^{\infty} h_k \boldsymbol{\Lambda}^k \tilde{\mathbf{x}} \qquad \blacksquare$$

- In the graph frequency domain graph filters are a diagonal matrices $\Rightarrow \tilde{\mathbf{y}} = \sum_{k=0}^{\infty} h_k \mathbf{\Lambda}^k \tilde{\mathbf{x}}$

- Thus, graph convolutions are pointwise in the GFT domain $\Rightarrow \tilde{y}_i = \sum_{k=0}^{\infty} h_k \lambda_i^k \tilde{x}_i = \tilde{h}(\lambda_i)\tilde{x}_i$

Definition (Frequency Response of a Graph Filter)

Given a graph filter with coefficients $\mathbf{h} = \{h_k\}_{k=1}^{\infty}$, the graph frequency response is the polynomial

$$\tilde{h}(\lambda) = \sum_{k=0}^{\infty} h_k \lambda^k$$

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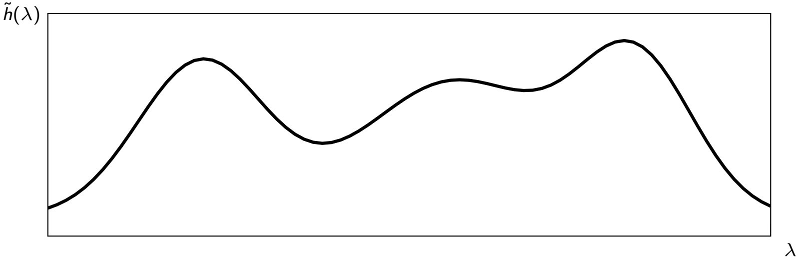
- Frequency response is the same polynomial that defines the graph filter $\Rightarrow$ but on scalar variable λ
- Frequency response is independent of the graph $\Rightarrow$ Depends only on filter coefficients
- The role of the graph is to determine the eigenvalues on which the response is instantiated

- Graph filter frequency response is a polynomial on a scalar variable λ $\Rightarrow \tilde{h}(\lambda) = \sum_{k=0}^{\infty} h_k \lambda^k$

- Completely determined by the filter coefficients $\mathbf{h} = \{h_k\}_{k=1}^{\infty}$. The Graph has nothing to do with it



The Graph Instantiates Specific Eigenvalues

- A given (another) graph instantiates the response on its given (different) specific eigenvalues λ_i
- Eigenvectors do not appear in the frequency response. They determine the meaning of frequencies.